The Shape of Central Quadrilaterals

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Abstract. The diagonals of a quadrilateral form four component triangles (in two ways). For each of various shaped quadrilaterals, we examine 1000 triangle centers located in these four component triangles. Using a computer, we determine when the four centers form a special quadrilateral, such as a rhombus or a cyclic quadrilateral. A typical result is the following. The diagonals of an equidiagonal quadrilateral divide the quadrilateral into four nonoverlapping triangles. Then the Nagel points of these four triangles form an orthodiagonal quadrilateral.

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1. Introduction

Consider the following two results.

Result 1

Let $ABCD$ be a cyclic quadrilateral. Then the incenters of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$ form a rectangle.

Result 2

Let $ABCD$ be a tangential quadrilateral. Then the incenters of the 4 nonoverlapping triangles formed by the diagonals lie on a circle.

At first glance, these two results seem unrelated. However, the following table points out their similarities.

Result	quadrilateral	component triangles	type of center	central quadrilateral
1	cyclic	half triangles	incenter	rectangle
2	tangential	quarter triangles	incenter	cyclic quadrilateral

The following definitions are used. A *cyclic quadrilateral* is a quadrilateral with a circumcircle. A *tangential quadrilateral* is a quadrilateral with an incircle. The diagonals of a convex quadrilateral divide it into four overlapping triangles each bounded by two sides of the quadrilateral and a diagonal. These are called *half triangles*. The diagonals of a convex quadrilateral divide it into four nonoverlapping triangles each bounded by a side of the quadrilateral and parts of the two diagonals. These are called *quarter triangles*. The *incenter* of a triangle is the center of the circle inscribed in the triangle. The four incenters form a quadrilateral called the *central quadrilateral*. The original quadrilateral is called the *reference quadrilateral*. Information about result 1 can be found in [1, p. 133]. Information about result 2 can be found in [5].

Looking at the table, we note the similarities. In both cases, we started with a quadrilateral with a special shape. We then considered four component triangles associated with the quadrilateral (half triangles or quarter triangles). Within each component triangle, we located a center, namely the incenter. We concluded that the four centers form the vertices of a quadrilateral that also have a special shape (rectangle or cyclic quadrilateral).

In this paper, we use a computer to discover similar results. In each case, we will start with a reference quadrilateral that has a special shape. We then form four component triangles related to that quadrilateral. Then we locate a triangle center in each component triangle. These four centers form a quadrilateral known as the *central quadrilateral*. Finally, we check to see if the central quadrilateral has a special shape.

2. Types of Quadrilaterals Studied

We are only interested in quadrilaterals that have a certain amount of symmetry. For example, we excluded bilateral quadrilaterals (those with two equal sides), bisect-diagonal quadrilaterals (where one diagonal bisects another), right kites, right trapezoids, and golden rectangles. The types of quadrilaterals we studied are shown in the following table. The sides of the quadrilateral, in order, have lengths a, b, c, and d. The measures of the angles of the quadrilateral, in order, are A, B, C, and D.

Types of Quadrilaterals Considered		
Quadrilateral Type	Geometric Definition	Algebraic Condition
general	convex	none
cyclic	has a circumcircle	$A + C = B + D$
tangential	has an incircle	$a + c = b + d$
extangential	has an excircle	$a + b = c + d$
parallelogram	opposite sides parallel	$a = c$, $b = d$
equalProdOpp	product of opposite sides equal	$ac = bd$
equalProdAdj	product of adjacent sides equal	$ab = cd$
orthodiagonal	diagonals are perpendicular	$a^2 + c^2 = b^2 + d^2$
equidiagonal	diagonals have the same length	
Pythagorean	equal sum of squares, adjacent sides	$a^2 + b^2 = c^2 + d^2$
kite	two pair adjacent equal sides	$a = b$, $c = d$
trapezoid	one pair of opposite sides parallel	$A + B = C + D$
rhombus	equilateral	$a = b = c = d$
rectangle	equiangular	$A = B = C = D$
Hjelmslev	two opposite right angles	$A = C = 90°$
isosceles trapezoid	trapezoid with two equal sides	$A = B$, $C = D$
APquad	sides in arithmetic progression	$d - c = c - b = b - a$

The following combinations of entries in the above list were also considered: bicentric quadrilaterals (cyclic and tangential), exbicentric quadrilaterals (cyclic and extangential), bicentric trapezoids, cyclic orthodiagonal quadrilaterals, equidiagonal orthodiagonal kites, equidiagonal orthodiagonal quadrilaterals, equidiagonal orthodiagonal trapezoids, harmonic quadrilaterals (cyclic and equalProdOpp), orthodiagonal trapezoids, tangential trapezoids, and squares (equiangular rhombi).

So, in addition to the general convex quadrilateral, a total of 27 other types of quadrilaterals were considered in this study. When checking to see if a central quadrilateral has one of these shapes, we also considered the following two degenerate quadrilaterals: a line segment (the four vertices of the quadrilateral are collinear) and a point (the four vertices coincide).

A graph of the types of quadrilaterals considered is shown in Figure 1. An arrow from A to B means that any quadrilateral of type B is also of type A. For example: all squares are rectangles and all kites are orthodiagonal.

3. Centers

In this study, we will place triangle centers in the four component triangles. We use Clark Kimberling's definition of a triangle center [7].

A *center function* is a nonzero function $f(a, b, c)$ homogeneous in a, b, and c and symmetric in b and c. Homogeneous in a, b, and c means that
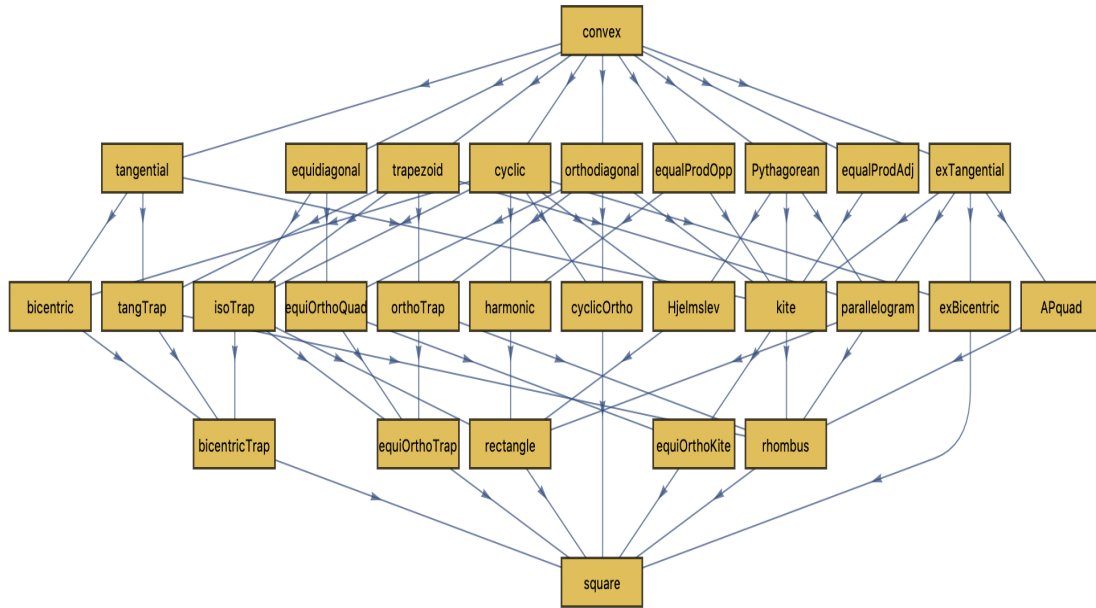
$$f(ta, tb, tc) = t^n f(a, b, c)$$

FIGURE 1. Quadrilateral Shapes

for some nonnegative integer n, all $t > 0$, and all positive real numbers (a, b, c) satisfying $a < b + c$, $b < c + a$, and $c < a + b$. Symmetric in b and c means that

$$f(a, c, b) = f(a, b, c)$$

for all a, b, and c.

A *triangle center* is an equivalence class $x : y : z$ of ordered triples (x, y, z) given by

$$x = f(a, b, c), \quad y = f(b, c, a), \quad z = f(c, a, b).$$

Tens of thousands of interesting triangle centers have been cataloged in the Encyclopedia of Triangle Centers [9]. We use X_n to denote the nth named center in this encyclopedia.

4. METHODOLOGY

We used a computer program called GeometricExplorer to determine the shape of the central quadrilateral. Starting with each type of quadrilateral listed in Figure 1 for the reference quadrilateral, and for each n from 1 to 1000, we placed center X_n in each of the component triangles of the reference quadrilateral. The program then analyzes the central quadrilateral formed by these four centers and reports if the central quadrilateral has a special shape. GeometricExplorer uses numerical coordinates (to 15 digits of precision) for locating all the points. This does not constitute a proof that the result is correct, but gives us compelling evidence for the validity of the result.

We then examine the center functions associated with each group of results and use that to guess at a pattern. Other center functions satisfying that pattern are then checked and if they too form the same shape central quadrilateral, then we list the result as a theorem in this paper. We then used exact symbolic computation using Mathematica to try and give a formal computer proof of the result. If a

proof is found, we include it with the supplementary material associated with this paper.

For example, using an arbitrary quadrilateral as the reference quadrilateral, when we place the center X_n in each of the quarter triangles formed by the diagonals, the resulting central quadrilateral appears to be a parallelogram for $n =$2, 3, 4, 5, 20, 140, 376, 382, 546, 547, 548, 549, 550, 631, and 632.

We then compiled a list of the center functions corresponding to each of these centers. There are multiple possibilities for each center function, since the center function is not unique. It can be scaled or expressed in various ways using either algebraic or trigonometric terms. The list is shown in Table 1.

Examining the entries in Table 1, we can make several conjectures. For example, we can conjecture that whenever the center function is of the form

$$\cos B \cos C + k \cos A$$

for some constant k, then the central quadrilateral will be a parallelogram. We test this conjecture by testing the center function $\cos B \cos C + \frac{7}{9} \cos A$ and GeometricExplorer returns the fact that the central quadrilateral is a parallelogram. We now have strong evidence that the result is true.

In this case, a formal computer proof was found and we list the result later on in this paper as Theorem 5.1.

TABLE 1.

n	center function
2	$\cos A + \cos(B - C)$
3	$\cos A$
4	$\cos A - \cos(B - C)$
5	$\cos(B - C)$
20	$\cos A - \cos B \cos C$
140	$2\cos A + \cos(B - C)$
376	$5\cos A + \cos(B - C)$
381	$2\cos(B - C) - \cos A$
382	$\cos A - 4\cos B \cos C$
546	$3\cos B \cos C - 2\cos A$
547	$5\cos B \cos C + 2\cos A$
548	$6\cos A - \cos(B - C)$
549	$\cos(B - C) + 4\cos A$
550	$-\cos(B - C) + 4\cos A$
631	$2\cos A + \cos B \cos C$
632	$7\cos A + 6\cos B \cos C$

If a theorem in this paper is accompanied by a figure, this means that the figure was drawn using either Geometer's Sketchpad or GeoGebra. In either case, we used the drawing program to dynamically vary the points in the figure. Noticing that the result remains true as the points vary offers further evidence that the theorem is true.

5. Results for Quarter Triangles

In this configuration, the reference quadrilateral is named $ABCD$. The two diagonals of the quadrilateral divide that quadrilateral into four small nonoverlapping triangles called quarter triangles. The point of intersection of the diagonals will be called E. The four triangles (numbered 1 to 4) are shown in Figure 2.
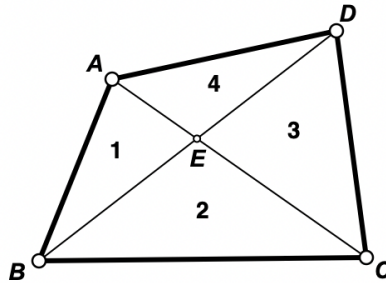

Figure 2. Quarter Triangles

The triangles have been numbered in a counterclockwise order starting with side AB: $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, $\triangle DAE$. Triangle centers are selected in each triangle. In order, their names are F, G, H, and I.

The raw data collected can be found in Appendix A. Looking for patterns in the raw data, we found a number of theorems and conjectures which are presented below.

5.1. General Quadrilaterals.

Theorem 5.1. *If the reference quadrilateral is a general quadrilateral, the central quadrilateral is a parallelogram whenever the chosen center has center function of the form*

$$\cos B \cos C + k \cos A$$

for some constant k.

Proof. Let $ABCD$ be the quadrilateral. Let E be the intersection of the diagonals. Without loss of generality, we can place B at the origin and C at the point with Cartesian coordinates $(1,0)$. Then let the coordinates of A, D, and E be as shown in Figure 3.
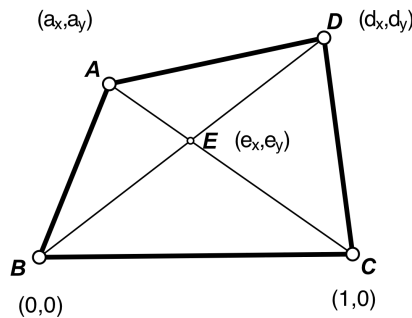

Figure 3. Cartesian coordinates for a general quadrilateral

Using the equations for lines AC and BD, we can find the coordinates of their intersection point, E. We get

$$E = \left(\frac{a_y d_x}{a_y d_x + d_y - a_x d_y}, \frac{a_y d_y}{a_y d_x + d_y - a_x d_y} \right).$$

Using the Law of Cosines, we can replace the cosines in the center function $\cos B \cos C + k \cos A$ to get an equivalent function in terms of a, b, and c. This gives us the 1st trilinear coordinate for the triangle center determined by this center function in terms of a, b, and c. Multiplying by a gives the 1st barycentric coordinate. We can clear fractions by multiplying the barycentric coordinates by the cyclic function $4abc$. We then find that the 1st barycentric coordinate for a point given by this center function is

$$a^4(1 - 2k) + 2a^2 k \left(b^2 + c^2\right) - \left(b^2 - c^2\right)^2.$$

Using the formula for the distance between two points in Cartesian coordinates, we can find the lengths of AB, BC, CD, DA, EA, EB, EC, and ED in terms of a_x, a_y, d_x, and d_y. For example,

$$DE = \frac{\left(a_y(d_x - 1) + d_y - a_x d_y\right)\sqrt{d_x^2 + d_y^2}}{a_y d_x + d_y - a_x d_y}.$$

If the lengths of the sides of a triangle are L_1, L_2, and L_3, and the coordinates of the vertices are $A = (a_x, a_y)$, $B = (b_x, b_y)$, and $C = (c_x, c_y)$, then the point with barycentric coordinates $(t_1 : t_2 : t_3)$ has Cartesian coordinates

$$(t_1 A + t_2 B + t_3 C)/(t_1 + t_2 + t_3)$$

where $tA = (ta_x, ta_y)$ and similarly for B and C (the change of coordinates formula).

We can thus find the Cartesian coordinates for point F, the center of $\triangle ABE$ corresponding to the given center function. We find that the coordinates of F are

$$\left(\frac{-a_x^2 d_y k + a_x(a_y d_x + d_y k) + a_y(d_x k + a_y d_y(1 - k))}{(2k + 1)(a_y d_x + d_y - a_x d_y)}, \right.$$
$$\left. \frac{a_x^2 d_x(k - 1) + a_x(-a_y d_y - d_x k + d_x) + a_y(a_y d_x k + d_y(k + 1))}{(2k + 1)(a_y d_x + d_y - a_x d_y)} \right).$$

In the same manner, we can find the coordinates for points G, H, and I, the centers of triangles $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$.

Using the point slope formula, we can find the slopes of FG, GH, HI, and IF. A little algebra shows that the slope of FG is the same as the slope of HI and the slope of GH is the same as the slope of IF. Thus $FG \parallel HI$ and $GH \parallel IJ$. Hence, the central quadrilateral $FGHI$ is a parallelogram. $\qquad\square$

These computations are straightforward, but very tedious, so best left to Mathematica. In the remainder of this paper, if a theorem is stated without a proof, we refer the reader to the Mathematica notebooks distributed as supplementary material to this paper. These notebooks contain the full proofs for many of the theorems in this paper.

Special Cases.

Common centers of the form given in Theorem 5.1 include the centroid (X_2), the circumcenter (X_3), the orthocenter (X_4), the nine-point center (X_5), and the

de Longchamps point (X_{20}). For some of these, simple geometric proofs can be found.

Corollary 5.2. *Let $ABCD$ be a convex quadrilateral with diagonal point E. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a parallelogram (Figure 4).*
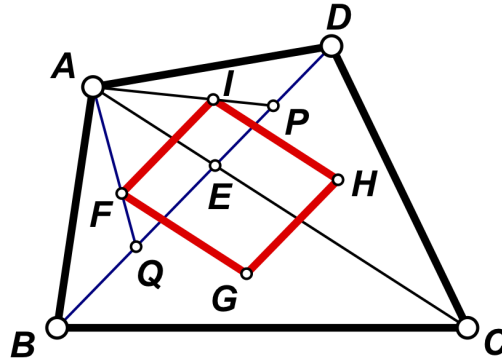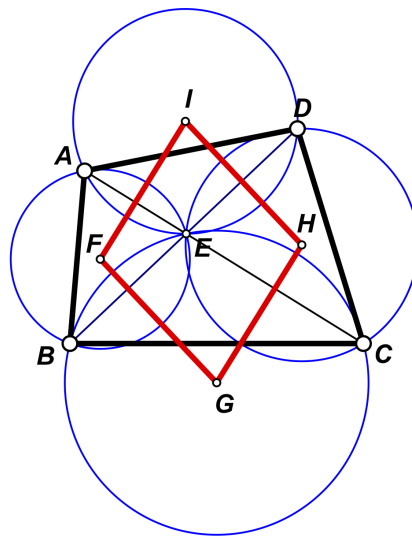

FIGURE 4. General quadrilateral: centroids $\implies$ parallelogram

Proof. Let AI meet BD at P and let AF meet BD at Q. Then AP is a median of $\triangle ADE$ and AQ is a median of $\triangle ABE$. Thus $AI = \frac{2}{3}AP$ and $AF = \frac{2}{3}AQ$ which implies $FI \parallel BD$. Similarly, $GH \parallel BD$. So $FI \parallel GH$. In the same way, $FG \parallel IH$. Hence, $FGHI$ is a parallelogram. $\square$

Corollary 5.3. *Let $ABCD$ be a convex quadrilateral with diagonal point E. Let F, G, H, and I be the circumcenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DEA$, respectively. Then $FGHI$ is a parallelogram (Figure 5).*
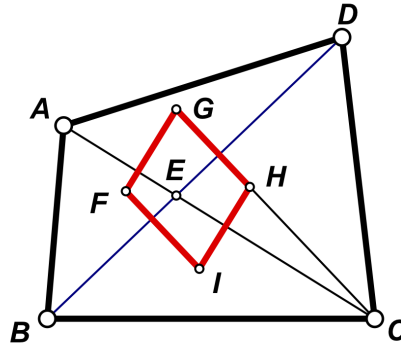

FIGURE 5. General quadrilateral: circumcenters $\implies$ parallelogram

Proof. The line of centers of two intersecting circles is perpendicular to the common chord. Thus $FG \perp BE$ and $HI \perp ED$. Thus $FG \parallel HI$. In the same manner, $GH \parallel FI$. Hence, $FGHI$ is a parallelogram. $\square$

Corollary 5.4. *Let $ABCD$ be a convex quadrilateral with diagonal point E. Let F, G, H, and I be the orthocenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a parallelogram (Figure 6).*

FIGURE 6. General quadrilateral: orthocenters $\implies$ parallelogram

Proof. Since CH is an altitude of $\triangle CED$, we must have $CH \perp ED$. Since CG is an altitude of $\triangle BEC$, we must have $CH \perp BE$. Thus, $GH \perp BD$. Similarly, $FI \perp BD$. Hence $GH \parallel FI$. In the same manner, $FG \parallel IH$. Therefore, $FGHI$ is a parallelogram. $\square$

Theorem 5.5. *All the points with center function of the form*

$$\cos B \cos C + k \cos A$$

lie on the Euler line.

Proof. The *Euler line* of a triangle is the line joining the centroid and the circumcenter. According to [9], a center function for the centroid G is $\cos B \cos C + \cos A$ and a center function for the circumcenter O is $\cos A$.

From [8, p. 28], the equation for the line joining $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ is

$$(1) \qquad \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

If P is the point with center function $\cos B \cos C + k \cos A$, then P lies on line OG because it is easy to verify that

$$\begin{vmatrix} \cos B \cos C + k \cos A & \cos C \cos A + k \cos B & \cos A \cos B + k \cos C \\ \cos B \cos C + \cos A & \cos C \cos A + \cos B & \cos A \cos B + \cos C \\ \cos A & \cos B & \cos C \end{vmatrix} = 0$$

because row 1 of the determinant is a linear combination of rows 2 and 3. $\square$

Using the same reasoning, we get the following result.

Theorem 5.6. *The center with center function p lies on the line joining the centers with center functions q and r if p is a linear combination of q and r.*

Theorem 5.7. *If the reference quadrilateral is a general quadrilateral, the central quadrilateral is a parallelogram whenever the chosen center has center function of the form*

$$\cos(B - C) + k \cos A$$

for some constant k.

Proof. From the trigonometric identity

$$2 \cos B \cos C = \cos(B - C) + \cos(B + C),$$

we see that

$$
\begin{aligned}
\cos(B - C) + k \cos A &= 2 \cos B \cos C - \cos(B + C) + k \cos A \\
&= 2 \cos B \cos C - \cos(180° - A) + k \cos A \\
&= 2 \cos B \cos C + \cos A + k \cos A \\
&= 2(\cos B \cos C + k' \cos A)
\end{aligned}
$$

where $k' = (k + 1)/2$. The result then follows from Theorem 5.1. $\square$

A function $f(a, b, c)$ is a *cyclic function* in a, b, and c if

$$f(a, b, c) = f(b, c, a) = f(c, a, b)$$

for all a, b, and c. Two center functions are said to be *equivalent* if their ratio is a cyclic function in a, b, and c. In particular, two center functions are equivalent if their ratio is a constant. The triangle centers corresponding to equivalent center functions coincide.

The proof of Theorem 5.7 shows that the center functions $\cos B \cos C + \frac{k+1}{2} \cos A$ and $\cos(B - C) + k \cos A$ are equivalent because

$$\frac{\cos B \cos C + \frac{k+1}{2} \cos A}{\cos(B - C) + k \cos A} = \frac{1}{2}$$

is identically true when A, B, and C are the angles of a triangle.

Conjecture 5.8. *If the reference quadrilateral is a general quadrilateral, the central quadrilateral is a parallelogram if and only if the center function is equivalent to $\cos B \cos C + k \cos A$ for some constant k.*

Theorem 5.9. *If the reference quadrilateral is a general quadrilateral, the central quadrilateral is orthodiagonal when the center is X_1. See Figure 7.*
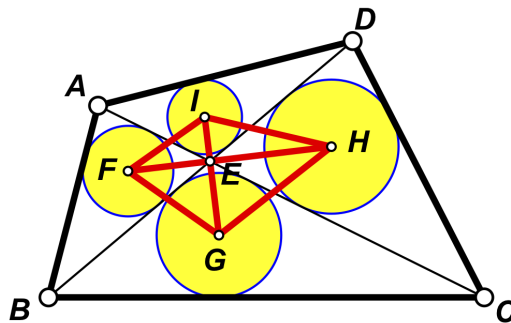

FIGURE 7. Central Quadrilateral is orthodiagonal

Proof. Since F is the incenter of $\triangle AEB$, this means that EF bisects $\angle AEB$. Similarly, EH bisects $\angle DEC$. Thus, FEH is a straight line. Similarly, IEG is a straight line. But $\angle FEG = \angle FEB + \angle BEG = \frac{1}{2}\angle AEB + \frac{1}{2}\angle BEC = \frac{1}{2}(180°) = 90°$ and so $FH \perp GI$. $\square$

Conjecture 5.10. *If the reference quadrilateral is a general quadrilateral, the central quadrilateral is orthodiagonal if and only if the center function is 1, that is, the center is X_1.*

5.2. **Cyclic Quadrilaterals.**

Theorem 5.11. *If the reference quadrilateral is cyclic, then the central quadrilateral is an isosceles trapezoid if the center is X_{110} (Figure 8).*
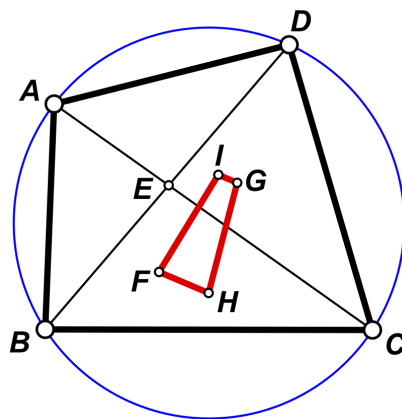
FIGURE 8. X_{110} points $\implies$ isosceles trapezoid

Note that in this case, the central quadrilateral is $FHGI$ as opposed to $FGHI$.

Theorem 5.12. *If the reference quadrilateral is cyclic, then $FH \perp GI$ when the center is one of the Vecten points, X_{485} or X_{486} (Figures 9 and 10).*

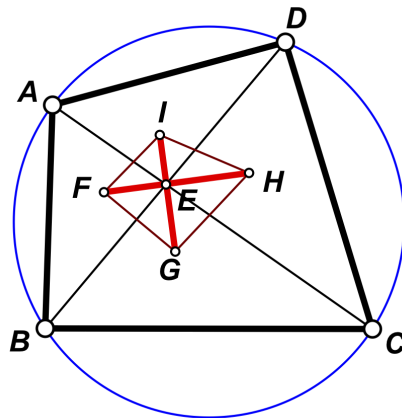**Note.** Figure 9 suggests that FH and GI pass through E. This is not the case.

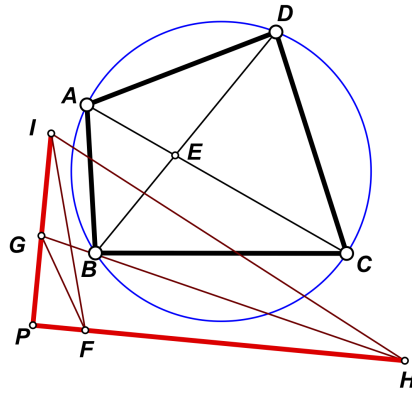FIGURE 9. X_{485} points $\implies$ $FH \perp GI$

FIGURE 10. X_{486} points $\implies FH \perp GI$

Proof. When dealing with cyclic quadrilaterals, it is more convenient to use barycentric coordinates (rather than Cartesian coordinates) because the equation for the circumcircle of a triangle in barycentric coordinates [13, p. 63] is very simple, namely

$$(2) \qquad a^2yz + b^2xz + c^2xy = 0.$$

We set up a barycentric coordinate system using $\triangle ABC$ as the reference triangle, so that $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$ and $C = (0 : 0 : 1)$. Let D have coordinates $(u : v : w)$. To make $ABCD$ convex, we will assume $u > 0$, $v < 0$, and $w > 0$. Line AC has equation $y = 0$ and line BD has equation $wx = uz$. Using the formula for the intersection of two lines, we can find the coordinates for point E, the intersection of the diagonals. See [4] for formulas involving barycentric coordinates. We get

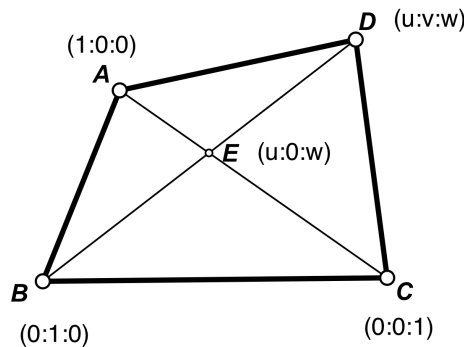$$E = (u : 0 : w).$$

The coordinate setup is shown in Figure 11.

FIGURE 11. Barycentric coordinates for a general quadrilateral

When D lies on the circumcircle of $\triangle ABC$, parametric equations for D are

$$u = c^2 + ta^2$$
$$v = -b^2 t$$
$$w = t(c^2 + ta^2)$$

where $0 < t < 1$. This can be verified by substituting these values into Equation 2.

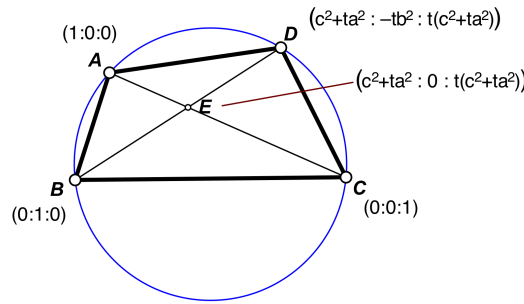This gives us the coordinate setup as shown in Figure 12.

FIGURE 12. Barycentric coordinates for a cyclic quadrilateral

Now we find the coordinates for F, G, H, and I. Instead of using the change of coordinates formula, we use the geometric definition of the outer Vecten point, since this only requires us to do some rotations through $90°$ and some intersections of lines. This avoids introducing square roots into our formulas. We get

$$F = \Big(-c^2 \left(a^2 t(t+3) + b^2 t(2t+3) + 4S(t+1) \right) +$$

$$t \left(a^4 - a^2 \left(b^2(t+2) + 2S(t+1) \right) + b^4(t+1) + 2b^2 S \right) + c^4 t(t+2) :$$
$$t \left(a^4(t+1) - a^2 \left(b^2(t+2) + 2c^2(t+1) \right) + b^4 - b^2 t \left(c^2 + 2S \right) + c^4(t+1) \right) :$$
$$t \left(a^4 t + a^2 \left(-2b^2 t - c^2(t-1) \right) + b^4 t - b^2 c^2(t+1) - c^2 \left(c^2 + 2S(t+1) \right) \right) \Big)$$

and similar expressions for G, H, and I, where S is twice the area of $\triangle ABC$. Finally, we use the formula that gives the condition for two lines to be perpendicular, and we find that $FH \perp GI$. These calculations are best done using Mathematica. A similar calculation provides a proof for the inner Vecten point. $\qquad \square$

5.3. **Tangential Quadrilaterals.**

Theorem 5.13. *If the reference quadrilateral is a tangential quadrilateral, then the central quadrilateral is cyclic if the chosen center has center function 1, i.e. the center is X_1. (Figure 13).*
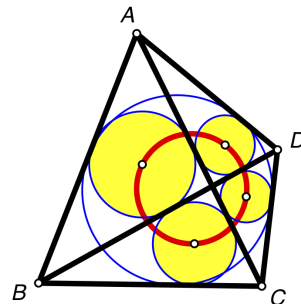

FIGURE 13. Tangential quadrilateral: incenters $\implies$ cyclic

Proof. This is Result 2 in the Introduction. For a geometric proof, see Theorem 20 of [3]. $\qquad \square$

Conjecture 5.14. *If the reference quadrilateral is a tangential quadrilateral, then the central quadrilateral is cyclic if and only if the center is X_1.*

5.4. **Equidiagonal Quadrilaterals.**

Theorem 5.15. *If the reference quadrilateral is equidiagonal, the central quadrilateral is a rhombus if the chosen center has center function of the form*

$$\cos B \cos C + k \cos A$$

for some constant k.

Theorem 5.15 is illustrated in Figure 14 when the chosen center is the orthocenter.
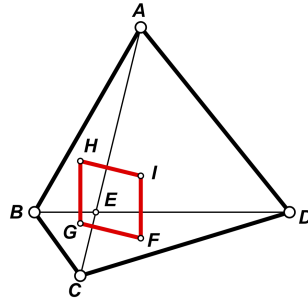

FIGURE 14. $AC = BD$, orthocenters $\implies$ rhombus

Corollary 5.16. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Let F, G, H, and I be the orthocenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a rhombus (Figure 14).*

Corollary 5.17. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a rhombus.*

For this corollary, a simple geometric proof can be given.

Proof. From the proof of Corollary 5.2 (Figure 4), it is easily seen that $IP = \frac{1}{3}AP$ and $FQ = \frac{1}{3}AQ$, so that $FI = \frac{1}{3}BD$. Similarly, $FG = \frac{1}{3}AC$. But since $BD = AC$, this implies $FI = FG$ and parallelogram $FGHI$ becomes a rhombus. $\square$

Corollary 5.18. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Let F, G, H, and I be the circumcenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a rhombus (Figure 15).*
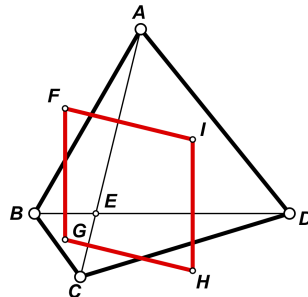

FIGURE 15. $AC = BD$, circumcenters $\implies$ rhombus

Theorem 5.19. *If the reference quadrilateral is equidiagonal, the central quadrilateral is orthodiagonal whenever the chosen center has center function of the form*

$$\frac{b+c}{a} + k.$$

Also, the diagonals of the central quadrilateral are parallel to the bimedians of the reference quadrilateral.

Note. A *bimedian* of a quadrilateral is the line joining the midpoints of two opposite sides.

Proof. When dealing with equidiagonal quadrilaterals, it is convenient to set up a coordinate system where the diagonal point of the quadrilateral is at the origin and one of the diagonals (AC) of the quadrilateral lies along the x-axis. We can scale the figure so that C lies at $(1, 0)$. Figure 16 shows the coordinates we use, where $a_x < 0$, $b_x \leq 0$, $b_y < 0$, $d_x \geq 0$, and $d_y > 0$.
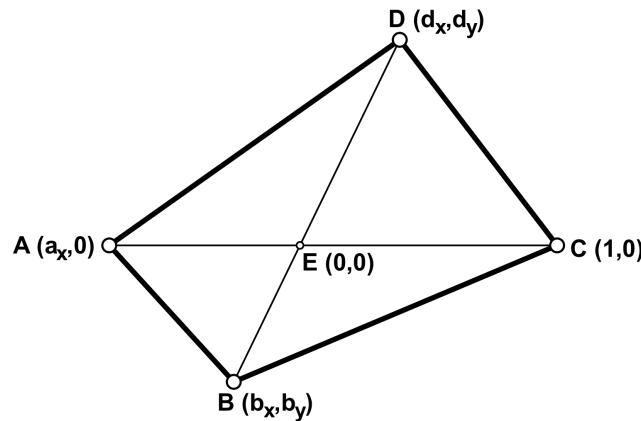

FIGURE 16. Cartesian Coordinates for an equidiagonal quadrilateral

In order for E to be at the origin, we need the slopes of BE and ED to be the same. We can ensure that this is the case by setting

$$d_y = \frac{d_x \cdot b_y}{b_x}.$$

In order for the quadrilateral to be equidiagonal, we need $AC = BD$ or equivalently $(AC)^2 = (BD)^2$. This gives the equation

$$(1 - a_x)^2 = \frac{(b_x^2 + b_y^2)(d_x - b_x)^2}{b_x^2}.$$

Solving for a_x and taking the solution that makes a_x negative, we find that we can ensure that the quadrilateral is equidiagonal by setting

$$a_x = 1 + \frac{(d_x - b_x)\sqrt{b_x^2 + b_y^2}}{b_x}.$$

Using $(b+c)/a+k$ as the center function, we can compute the coordinates for F, G, H, and I. Using the formula for the slope of a line through two points, we find the slopes of FH and GI. Then (using Mathematica) we find and simplify the product of these slopes. The result is -1, proving that $FGHI$ is orthodiagonal. □

Since the Spieker center has center function $(b+c)/a$, it has the form referenced by Theorem 5.19 (with $k = 0$), so we have the following corollary.

Corollary 5.20. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Let F, G, H, and I be the Spieker centers of $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then quadrilateral $FGHI$ is orthodiagonal. (Figure 17)*
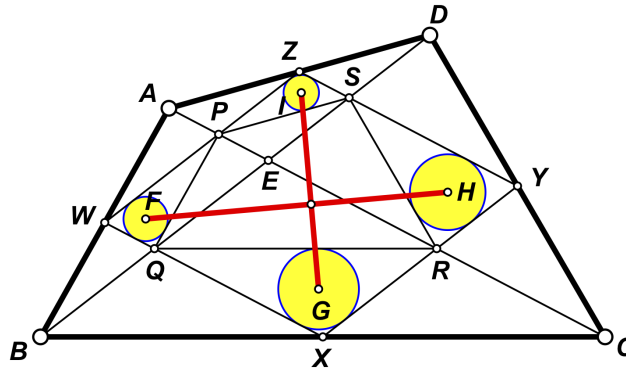

FIGURE 17. Equidiagonal quadrilateral: Spieker centers $\implies$ orthodiagonal

In this case, we have a simple purely geometric proof.

Proof. Let the midpoints of the sides of the quadrilateral be W, X, Y, and Z as shown. Let the Spieker centers be F, G, H, and I. Let P, Q, R, and S be the midpoints of AE, BE, CE, and DE, respectively. The Spieker center of a triangle is the incenter of its medial triangle. Thus, F, G, H, and I are the centers of the yellow circles in Figure 17. Since Y, S, and Z are midpoints, YSZ is a straight line segment and $YZ = \frac{1}{2}AC$. Similarly, $ZW = \frac{1}{2}BD$, $WX = \frac{1}{2}AC$, and $XY = \frac{1}{2}BD$.

Since quadrilateral $ABCD$ is equidiagonal, quadrilateral $WXYZ$ is a rhombus. The diagonals of a rhombus are perpendicular, so $WY \perp XZ$. The diagonals of a rhombus also bisect the angles at the vertices. Since H is the incenter of $\triangle SYR$, YH is the angle bisector of $\angle SYR$. Since YW is also the angle bisector of $\angle SYR$, we can conclude that WY passes through H and F. In the same manner, ZX passes through I and G. Since $WY \perp XZ$, we can conclude that $GI \perp FH$ or that quadrilateral $FGHI$ is orthodiagonal.

Note that in this case, the diagonals of $FGHI$ coincide with the bimedians of $ABCD$. $\square$

Since the Nagel point has center function $(b+c)/a - 1$, we have the following corollary.

Corollary 5.21. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Let F, G, H, and I be the Nagel points of $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FH \perp GI$. In other words, quadrilateral $FGHI$ is orthodiagonal. Furthermore, FH and GI are parallel to the bimedians of quadrilateral $ABCD$. (Figure 18)*
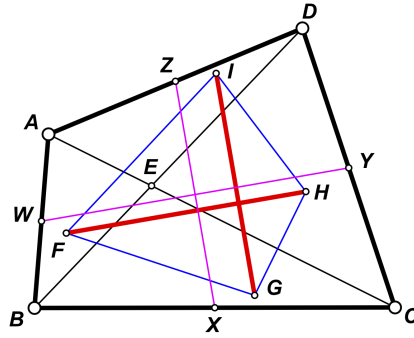
FIGURE 18. Equidiagonal quadrilateral: Nagel points $\implies$ orthodiagonal

Open Question 1. *Is there a simple geometric proof for the special case of Theorem 5.19 when the chosen center is the Nagel point? (Figure 18) Is there a geometrical proof that in Figure 18, $FH \parallel WY$?*

Theorem 5.22. *All the points with center function of the form*

$$\frac{b+c}{a} + k$$

lie on the Nagel line.

Proof. The *Nagel line* of a triangle is the line joining the incenter, the centroid, the Nagel point, and the Spieker center of a triangle. From [9], we find that a center function for the incenter is $I = 1$ and a center function for the Spieker center is $S = (b+c)/a$. Let $P = (b+c)/a + k$. Since $P = S + kI$, P is a linear combination of I and S and our result follows from Theorem 5.6. $\square$

Theorem 5.23. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Let F, G, H, and I be the incenters of $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then FH and GI are parallel to the bimedians of quadrilateral $ABCD$ (Figure 19).*
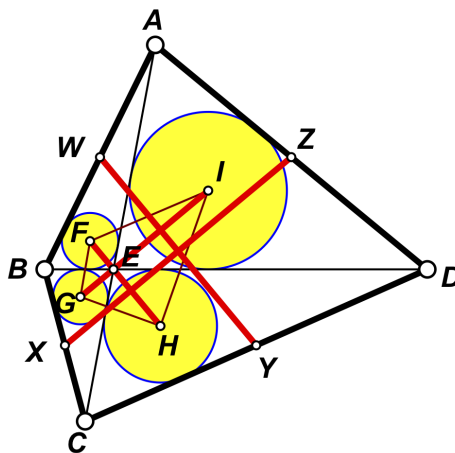

FIGURE 19. $AC = BD$, incenters, bimedians $\implies FH \parallel WY$

Theorem 5.24. *If the reference quadrilateral is equidiagonal, the central quadrilateral is orthodiagonal whenever the chosen center has center function of the form*

$$\cos B + \cos C - 1 - k \cos A.$$

Since the Bevan point has center function $\cos B + \cos C - 1 - \cos A$, we have the following corollary.

Corollary 5.25. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Let F, G, H, and I be the Bevan points of $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FH \perp GI$ (Figure 20).*
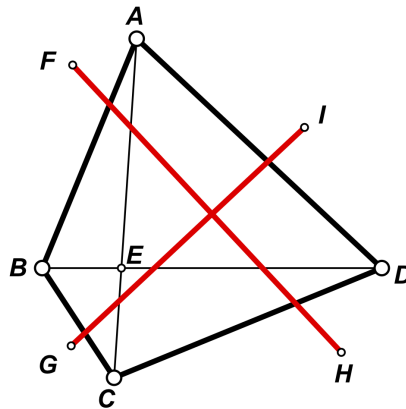

FIGURE 20. $AC = BD$, Bevan points $\implies$ $FH \perp GI$

Theorem 5.26. *All the points with center function of the form*

$$\cos B + \cos C - 1 - k \cos A$$

lie on the Bevan line.

Proof. The *Bevan line* of a triangle (also known as the OI line) is the line joining the incenter, the circumcenter, and the Bevan point. From [9], we find that a center function for the circumcenter is $O = \cos A$ and a center function for the Bevan point is $B = \cos B + \cos C - 1 - \cos A$. Let $P = \cos B + \cos C - 1 - k \cos A$. Since $P = B - (k-1)O$, P is a linear combination of O and P and our result follows from Theorem 5.6. $\qquad\square$

Theorem 5.27. *If the reference quadrilateral is equidiagonal, the central quadrilateral is orthodiagonal whenever the chosen center has center function of the form*

$$\frac{r}{R} + k \cos A.$$

Proof. Using the rules

(3)
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$R = \frac{abc}{4K}$$
$$r = \frac{K}{s}$$
$$K = \sqrt{s(s-a)(s-b)(s-c)}$$
$$s = \frac{a+b+c}{2}$$

we can remove A, B, C, r, and R from the expression

$$\frac{\cos B + \cos C - 1 - k \cos A}{\frac{r}{R} + t \cos A}.$$

When $t = -(k+1)$, the expression simplifies to 1. This means that

$$\cos B + \cos C - 1 - k \cos A \quad \text{and} \quad \frac{r}{R} + t \cos A$$

are equivalent center functions. Thus, this theorem follows from Theorem 5.24. □

Theorem 5.28. *If the reference quadrilateral is equidiagonal, the central quadrilateral is orthodiagonal whenever the chosen center has center function of the form*

$$\frac{r}{R} + k \sin B \sin C.$$

5.5. **Orthodiagonal Quadrilaterals.**

Lemma 5.29. *The condition for a triangle center with center function $f(x, y, z)$ to lie on the angle bisector at vertex A in right triangle ABC (with right angle at A) is*

$$f(x, y, z) = f(y, x, z)$$

for all x, y, and z satisfying $x^2 + y^2 = z^2$.

Proof. Let P be a point on the angle bisector of $\angle A$ in right triangle ABC as shown in Figure 21. Suppose a center function for P is $f(x, y, z)$ so that the trilinear coordinates of P are

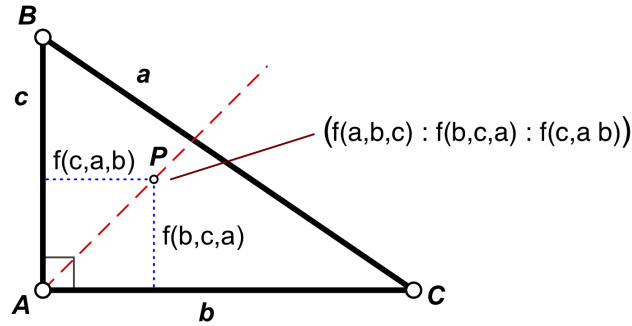$$P = \big(f(a, b, c) : f(b, c, a) : f(c, a, b)\big).$$

FIGURE 21. P lies on bisector of $\angle A$ in right triangle ABC

The second trilinear coordinate is proportional to the distance from P to AC and the third trilinear coordinate is proportional to the distance from P to AB. Since P is on the angle bisector of $\angle A$, it is equidistant from AC and AB. Thus,

$$f(b, c, a) = f(c, a, b)$$

for all a, b, and c satisfying $a^2 = b^2 + c^2$. But, by definition, a center function is symmetric in its 2nd and 3rd arguments, so

$$f(c, a, b) = f(c, b, a)$$

for all a, b and c. Therefore,

$$f(b, c, a) = f(c, b, a)$$

for all a, b and c satisfying $a^2 = b^2 + c^2$. Switching notation from a, b, c to x, y, z gives that the condition is that $f(x, y, z) = f(y, x, z)$ for all x, y, and z satisfying $x^2 + y^2 = z^2$. $\qquad\square$

Theorem 5.30. *Let $f(x, y, z)$ be a polynomial center function such that*

$$(4) \qquad f(a, b, c) - f(b, a, c) \quad has \quad a^2 + b^2 - c^2 \quad as \ a \ factor.$$

If the reference quadrilateral is orthodiagonal, then the central quadrilateral is orthodiagonal whenever the chosen center has center function f.

Proof. If Q is the center corresponding to $\triangle AEB$, then by Lemma 5.29, EQ bisects $\angle AEB$. Similarly, if P is the center corresponding to $\triangle AED$, then EP bisects $\angle AED$ (Figure 22).
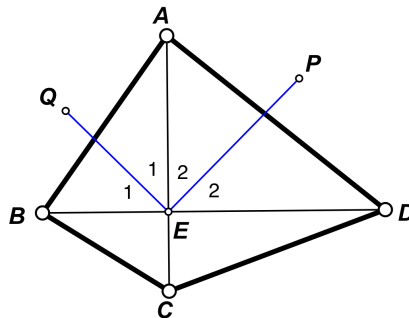

FIGURE 22. orthodiagonal quadrilateral with two angle bisectors

Then we see that angles 1 and 2 both have measure $45°$, so $EQ \perp EP$. Thus, the central quadrilateral is orthodiagonal and has the same diagonal point as the reference quadrilateral. $\square$

Let us call a function f that has the form specified by Theorem 5.30 a *normal center function*. We may also say that f is *normal*.

The following lemma should be obvious.

Lemma 5.31. *If f is normal, then so is $f + k$, where k is a constant. If f is normal, then so is kf, where k is a constant. If f is normal, then so is f^n, where n is a positive integer. If f and g are normal, then so is $f + g$.*

Proof. Let $g = f + k$. Then $g(a,b,c) - g(b,a,c) = f(a,b,c) - f(b,a,c)$. Let $E = kf$. Then $E(a,b,c) - E(b,a,c) = k(f(a,b,c) - f(b,a,c))$. If $h = f^n$, note that $h(a,b,c) - h(b,a,c) = f(a,b,c)^n - f(b,a,c)^n$ is divisible by $f(a,b,c) - f(b,a,c)$ which is divisible by $a^2 + b^2 - c^2$. If $F = f + g$, note that $F(a,b,c) - F(b,a,c) = f(a,b,c) - f(b,a,c) + g(a,b,c) - g(b,a,c)$ which is divisible by $a^2 + b^2 - c^2$. $\square$

Theorem 5.32. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is orthodiagonal whenever the chosen center has center function of the form*

$$\cos B \cos C + k$$

for some constant k.

Proof. By Lemma 5.31, it suffices to prove this for center function $\cos B \cos C$. Applying transformation set (3), we find that $\cos B \cos C$ is equivalent to the center function

$$\frac{a^4 - (b^2 - c^2)^2}{4a^2bc}.$$

Multiplying this by the cyclic function $4a^2b^2c^2$ shows this to be equivalent to the center function

$$f(a,b,c) = bc(a^4 - (b^2 - c^2)^2).$$

A straightfoward calculation shows that

$$f(a,b,c) - f(b,a,c) = c(a-b)(a+b-c)(a+b+c)\left(a^2 + b^2 - c^2\right),$$

which has the factor $a^2 + b^2 - c^2$. Thus, f is normal and the central quadrilateral is orthodiagonal by Theorem 5.30. $\square$

Theorem 5.33. *Let $f(x,y,z)$ be a polynomial center function such that*

(5) $\qquad f(a,b,c) + f(b,a,c) \quad has \quad a^2 + b^2 - c^2 \quad as\ a\ factor.$

If the reference quadrilateral is orthodiagonal, then the central quadrilateral is orthodiagonal whenever the chosen center has center function f.

Proof. The proof is the same as the proof of Theorem 5.30 except that a variant of Lemma 5.29 is used to show when a point lies on the external angle bisector of the right angle in a right triangle. The details are omitted. $\square$

Using the same proof technique that was used in the proof of Theorem 5.32, combined with either Theorem 5.30 or Theorem 5.33, we can prove the following theorem.

Theorem 5.34. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is orthodiagonal whenever the chosen center has center function of one of the following forms.*

$$\sec A$$
$$\sec B$$
$$\tan A$$
$$\sin 2A$$
$$\cos B \cos C$$
$$\tan B + \tan C$$
$$\sec B + \sec C$$
$$\sec B - \sec C$$
$$\cos 2A$$
$$\cos B + \cos C - \cos A$$
$$1/(\cos B + \cos C - \cos A)$$
$$\sec 2A$$
$$\sin 4A$$
$$\tan B$$
$$\tan B - \tan C$$

Using these forms combined with Lemma 5.31 shows that the following centers produce orthodiagonal central quadrilaterals when the given quadrilateral is orthodiagonal.

A selection of centers that make the central quadrilateral orthodiagonal	
center	center function
X_{46}	$\cos B + \cos C - \cos A$
X_{47}	$\cos 2A$
X_{48}	$\tan B + \tan C$
X_{73}	$\sec B + \sec C$
X_{90}	$1/(\cos B + \cos C - \cos A)$
X_{91}	$\sec 2A$
X_{223}	$\sec B + \sec C - \sec A - 1$
X_{388}	$\cos B \cos C + 1$
X_{497}	$\cos B \cos C - 1$
X_{563}	$\sin 4A$
X_{610}	$\tan B + \tan C - \tan A$
X_{652}	$\sec B - \sec C$
X_{656}	$\tan B - \tan C$
X_{822}	$(\sec B + \sec C)(\sec B - \sec C)$

Theorem 5.35. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is equidiagonal if the chosen center is X_{151}.*

Theorem 5.36. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is cyclic if the chosen center is the symmedian point (X_6).*

Proof. When dealing with orthodiagonal quadrilaterals, it is convenient to set up a coordinate system where the diagonal point of the quadrilateral is at the origin and the diagonals of the quadrilateral lie along the x- and y-axes. We can scale the figure so that vertex D lies at $(1,0)$. Figure 23 shows the coordinates we use, where $a_y > 0$, $b_x < 0$, and $c_y < 0$.

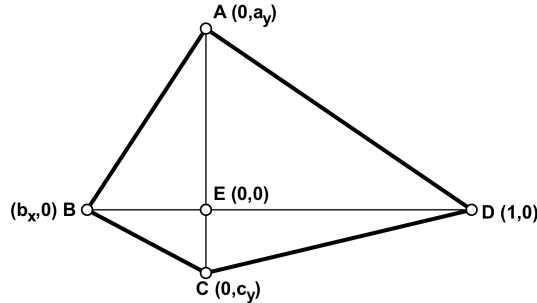FIGURE 23. Cartesian coordinates for an orthodiagonal quadrilateral

Using a as the center function for X_6, we can compute the coordinates for F, G, H, and I. We find that the coordinates for F are

$$F = \left(\frac{b_x a_y^2}{2(a_y^2 + 2b_x^2)}, \frac{a_y a b_x^2}{2(a_y^2 + 2b_x^2)} \right).$$

Similar expressions are found for the coordinates for G, H, and I. Then we find the coordinates for P, the intersection of FH and GI. We find

$$P = \left(\frac{b_x a_y^2 c_y^2 (b_x + 1)}{2 \left(b_x^2 c_y^2 + a_y^2 c_y^2 + a_y^2 b_x^2 (1 + c_y^2) \right)}, \frac{a_y b_x^2 c_y (a_y + c_y)}{2 \left(b_x^2 c_y^2 + a_y^2 c_y^2 + a_y^2 b_x^2 (1 + c_y^2) \right)} \right).$$

Using the distance formula, we find the lengths of PF, PG, PH, and PI. The points F, G, H, and I will lie on a circle if $PF \cdot PH = PG \cdot PI$ (the converse of the intersecting chords theorem for a circle). A straightforward computation (using Mathematica) shows this to be true. Hence $FGHI$ is a cyclic quadrilateral. $\square$

Open Question 2.
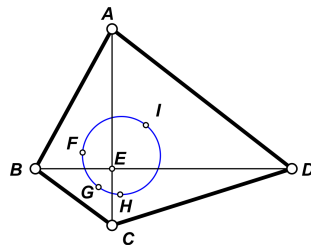*Is there a purely geometric proof for Theorem 5.36? (Figure 24)*

FIGURE 24. Orthodiagonal quadrilateral: symmedian points $\implies$ cyclic

Theorem 5.37. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is cyclic whenever the chosen center has center function of the form*

$$k \cos A + \sin A (\tan A + \tan B + \tan C)$$

for some constant k.

Note. Examples of centers that have center functions of this form are X_{389} and X_{578}.

Theorem 5.38. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is a rectangle whenever the chosen center has center function of the form*

$$\cos A + k \cos B \cos C$$

where k is a constant.

For some special cases of this theorem, a simpler geometrical proof is possible.

The centroid has center function $\cos A + \cos B \cos C$. This is of the form referenced by Theorem 5.38 when $k = 1$. We thus have the following corollary.

Corollary 5.39. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is a rectangle when the chosen center is the centroid (Figure 25).*
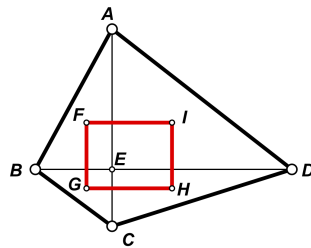
FIGURE 25. Orthodiagonal quadrilateral: centroids $\implies$ rectangle

Proof. By Theorem 5.2, $FGHI$ is a parallelogram. From the proof of Theorem 5.2, we see that $FI \parallel BD$ and $FG \parallel AC$. Since $BD \perp AC$, this implies $IF \perp FG$. But a parallelogram having one angle a right angle must be a rectangle. $\qquad\square$

The circumcenter has center function $\cos A$. This is of the form referenced by Theorem 5.38 when $k = 0$. We thus have the following corollary.

Corollary 5.40. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is a rectangle when the chosen center is the circumcenter (Figure 26).*
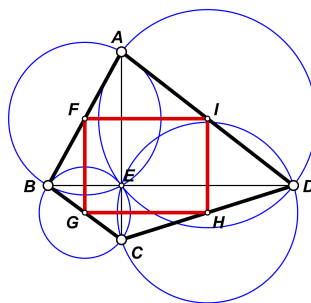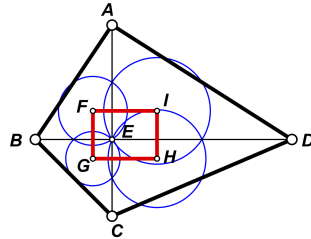

FIGURE 26. Orthodiagonal quadrilateral: circumcenters $\implies$ rectangle

Proof. By Theorem 5.3, $FGHI$ is a parallelogram. From the proof of Theorem 5.3, we see that $FI \perp AC$ and $FG \perp BD$. Since $BD \perp AC$, this implies $IF \perp FG$. But a parallelogram having one angle a right angle must be a rectangle. $\qquad\square$

We could also have noted that the circumcenter of a right triangle is the midpoint of the hypotenuse and used that fact to prove that the sides of the central quadrilateral are parallel to the diagonals of the reference quadrilateral.

The nine-point center has center function $\cos A + 2 \cos B \cos C$. This is of the form referenced by Theorem 5.38 when $k = 2$. We thus have the following corollary.

Corollary 5.41. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is a rectangle when the chosen center is the nine-point center. (Figure 27).*

FIGURE 27. Orthodiagonal quadrilateral: nine-point centers $\implies$ rectangle

Proof. Since $\odot F$ is the nine-point circle of right triangle AEB, it passes through the foot of the altitude from B to AE, which is point E. It also passes through the midpoint of AE. In the same way, $\odot I$ passes through E and the midpoint of AE. Thus, AE is the common chord between these two circles which implies that the line of centers, FI is perpendicular to AC. Similarly, $FG \perp BD$, $GH \perp AC$, and $HI \perp BD$. Since $AC \perp BD$, this implies that $FG \perp GH$, $GH \perp HI$, $HI \perp IF$, and $IF \perp FG$, making $FGHI$ a rectangle. $\qquad\square$

5.6. **Hjelmslev Quadrilaterals.**

Theorem 5.42. *If the reference quadrilateral is a Hjelmslev quadrilateral, the central quadrilateral is orthodiagonal whenever the chosen center has center function of the form*

$$\tan(2B) + \tan(2C) + k \tan(2A)$$

for some constant k.

Theorem 5.43. *If the reference quadrilateral is a Hjelmslev quadrilateral, then the central quadrilateral is a trapezoid when the chosen center is the Prasolov point (X_{68}). See Figure 28.*
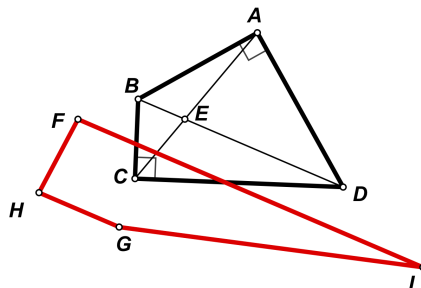

FIGURE 28. Prasolov points $\implies$ $FI \parallel HG \parallel BD$

5.7. **Parallelograms.**

Lemma 5.44 (The Parallelogram Lemma). *Let $ABCD$ be a parallelogram and let E be the intersection of the diagonals. Let P be a triangle center of $\triangle BEC$ and let Q be the corresponding triangle center of $\triangle DEA$. Then PQ passes through E and $PE = QE$. (Figure 29)*
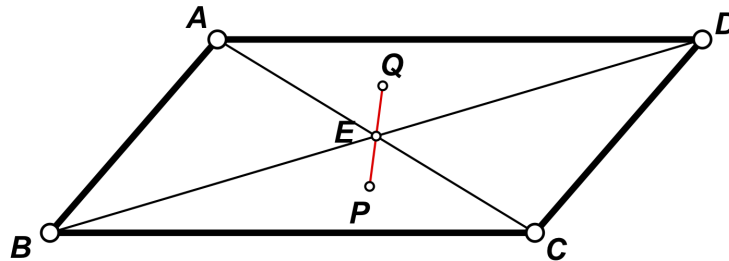
FIGURE 29.

Proof. Note that triangles BEC and DEA are congruent. Under the congruence transformation that maps $\triangle BEC$ into $\triangle DEA$, P will get mapped into Q. Since E is the center of the congruence transformation, this means PQ will pass through E. Congruence preserves lengths, so $PE = QE$.   □

Theorem 5.45. *For any triangle center, if the reference quadrilateral is a parallelogram, then the central quadrilateral is a parallelogram. The sides of the central quadrilateral are parallel to the diagonals of the reference quadrilateral. The two quadrilaterals have the same diagonal point. (Figure 30)*
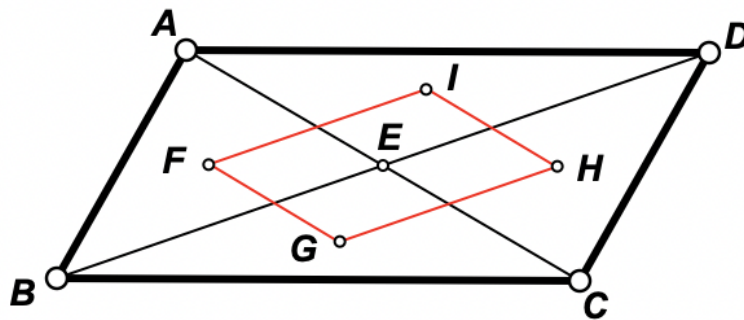
FIGURE 30. parallelogram $\implies$ parallelogram

Proof. By the Parallelogram Lemma, E is the midpoint of GI. Similarly, E is the midpoint of FH. Since the diagonals of quadrilateral $FGHI$ bisect each other, $FGHI$ must be a parallelogram.   □

Theorem 5.46. *If the reference quadrilateral is a parallelogram, the central quadrilateral is a rhombus if the chosen center is the incenter (X_1) or one of the Vecten points (X_{485} or X_{486}).*

5.8. **Kites.**

Lemma 5.47 (The Isosceles Triangle Lemma). *Let ABC be an isosceles triangle (with $AB = AC$) and let D be the midpoint of BC. Let P be a triangle center of $\triangle ABD$ and let Q be the corresponding triangle center of $\triangle ACD$. Then $PQ||BC$, $PQ \perp AD$, and PQ is bisected by AD.*
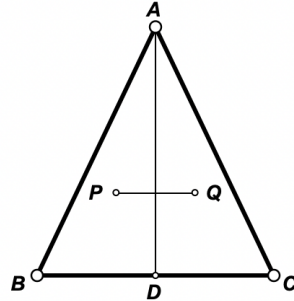
Proof. Note that $AD \perp BC$. Under the congruence transformation that maps $\triangle ABD$ into $\triangle ACD$, Q will be the reflection of P about AD (and PQ is bisected by AD). Thus $PQ \perp AD$ which implies $PQ||BC$. $\qquad\square$

Theorem 5.48. *For any triangle center, if the reference quadrilateral is a kite, then the central quadrilateral is an isosceles trapezoid. The parallel sides of the trapezoid are parallel to one of the diagonals of the reference quadrilateral and are bisected by the other diagonal (Figure 31).*
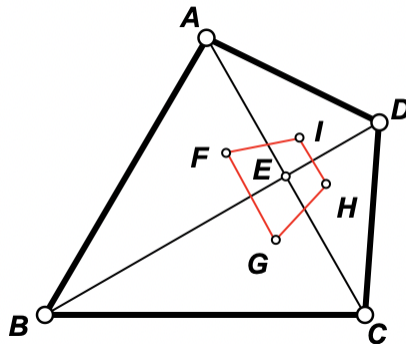

FIGURE 31. kite $\implies$ isosceles trapezoid

Proof. Triangle DAC is isosceles, so by the Isosceles Triangle Lemma (Lemma 5.47), $HI||AC$. Similarly, $FG||AC$, so $FGHI$ is a trapezoid. Also by the Isosceles Triangle Lemma, ED bisects HI. From the fact that H and I are equidistant from BD, and F and G are equidistant from BD, we can conclude that $GH = FI$ and the trapezoid is isosceles. $\qquad\square$

Theorem 5.49. *If the reference quadrilateral is a kite, the central quadrilateral is a square if the chosen center is the incenter (X_1) or one of the Vecten points (X_{485} or X_{486}).*

Note. There are many other centers for which the central quadrilateral is a square.

5.9. **Rhombi.**

Theorem 5.50. *For any triangle center, if the reference quadrilateral is a rhombus, then the central quadrilateral is a rectangle. The two quadrilaterals have the same diagonal point. The sides of the rectangle are parallel to diagonals of the rhombus and are bisected by them (Figure 32).*
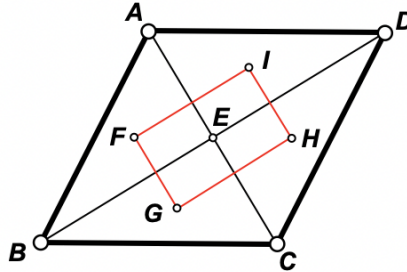

FIGURE 32. rhombus $\implies$ rectangle

Proof. By Theorem 5.48, the central quadrilateral must be an isosceles trapezoid. By Theorem 5.45, the central quadrilateral must be a parallelogram. But a quadrilateral that is both an isosceles trapezoid and a parallelogram must be a rectangle. □

5.10. **Isosceles Trapezoids.**

Theorem 5.51. *For any triangle center, if the reference quadrilateral is an isosceles trapezoid, then the central quadrilateral is a kite. The two quadrilaterals have the same diagonal point. One diagonal is parallel to the parallel sides of the trapezoid. (Figure 33)*
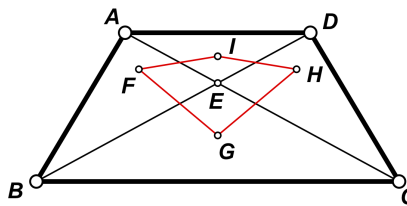

FIGURE 33. isosceles trapezoid $\implies$ kite

Proof. Note that $\triangle AED$ and $\triangle BEC$ are isosceles triangles. All triangle centers of an isosceles triangle lie on the altitude to the base. We can conclude that G and I lie on the perpendicular bisector of BC (which passes through E). Now note that $\triangle AEB \cong \triangle DEC$. One triangle is the mirror image of the other when reflected about the perpendicular bisector of BC. Under the reflection that maps $\triangle AEB$ into $\triangle DEC$, F will get mapped into H. Thus, GI is the perpendicular bisector of FH. Hence, $FI = HI$ and $FG = HG$ which means that quadrilateral $FGHI$ is a kite. □

5.11. **Rectangles.**

Theorem 5.52. *For any triangle center, if the reference quadrilateral is a rectangle, then the central quadrilateral is a rhombus. The two quadrilaterals have the same diagonal point. The sides of the rhombus are parallel to diagonals of the rectangle and are bisected by them (Figure 34).*
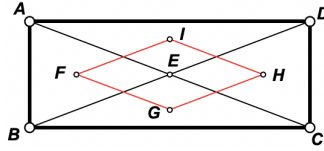

FIGURE 34. rectangle $\implies$ rhombus

Proof. A rectangle can be considered to be an isosceles trapezoid in two different ways. Thus, this result follows from Theorem 5.51 since both quadrilaterals $FGHI$ and $GHIF$ are kites. $\square$

5.12. **Squares.**

Theorem 5.53. *For any triangle center, if the reference quadrilateral is a square, then the central quadrilateral is a square (Figure 35).*
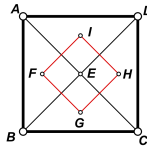

FIGURE 35. square $\implies$ square

Proof. By Theorem 5.52, the central quadrilateral must be a rhombus. By Theorem 5.50, the central quadrilateral must be a rectangle. But a quadrilateral that is both a rhombus and a rectangle must be a square. $\square$

Theorem 5.54. *Let $ABCD$ be a square with diagonal point E. Let F be the reflection of the incenter of $\triangle ABE$ about the Feuerbach point of $\triangle ABE$ (X_{80}). Define G, H, and I similarly using triangles $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$. Then quadrilateral $FGHI$ is congruent to $ABCD$.*

The same result holds for the Bevan point (X_{40}) and its isogonal conjugate (X_{84}). See Figure 36. This is also an example where the vertices of the reference quadrilateral and the central quadrilateral lie on the same circle.
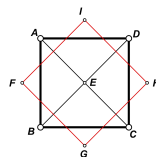

FIGURE 36. square: Bevan points $\implies$ congruent square

6. RESULTS FOR HALF TRIANGLES

In this configuration, the reference quadrilateral is named $ABCD$. Each diagonal of the quadrilateral divides the quadrilateral into two triangles. Since there are two diagonals, a total of four triangles are formed. The four triangles (numbered 1 to 4) are shown in Figure 37.
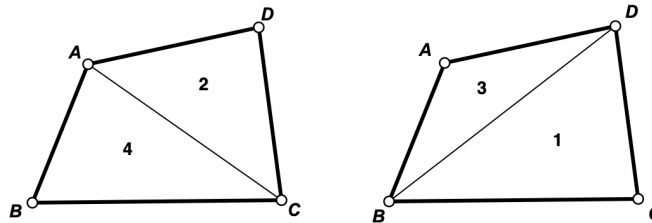

FIGURE 37. Half Triangles

The triangles have been numbered so that triangle 1 is opposite vertex A, triangle 2 is opposite vertex B, etc. The four triangles are $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$. Triangle centers are selected in each triangle. In order, their names are E, F, G, and H.

The raw data collected can be found in Appendix B. Looking for patterns in the raw data, we found a number of theorems and conjectures which are presented below.

6.1. **General Quadrilaterals.**

Conjecture 6.1. *If the reference quadrilateral is a general quadrilateral, $ABCD$, there is no center function such that the central quadrilateral is a cyclic quadrilateral for all quadrilaterals $ABCD$.*

Conjecture 6.2. *If the reference quadrilateral is a general quadrilateral, $ABCD$, there is no center function such that the central quadrilateral is a rectangle for all quadrilaterals $ABCD$.*

Theorem 6.3. *If the reference quadrilateral is a general quadrilateral and the chosen center is X_2, then the central quadrilateral is similar to the reference quadrilateral. The ratio of similitude is 3 (Figure 38).*
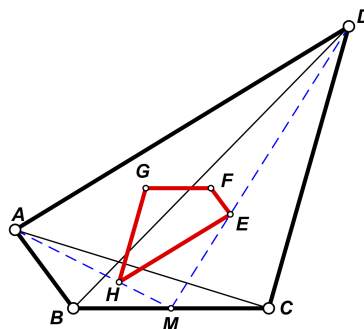

FIGURE 38. centroids $\implies EFGH \sim ABCD$

Proof. Let M be the midpoint of BC. Then AM and DM are medians and $DM/EM = 3$ and $AM/HM = 3$. Therefore, in $\triangle AMD$, we must have $HE \parallel AD$ and $AD/HE = 3$. Similarly, we have the same ratio and parallelism for EF, FG, and GH. Thus, the sides of quadrilateral $ABCD$ are 3 times the corresponding sides of quadrilateral $EFGH$. The parallelism implies that the angles of the two quadrilaterals are equal. Since the sides are in proportion and the angles are equal, the two quadrilaterals must be similar.   □

Theorem 6.4. *If the reference quadrilateral is a general quadrilateral and the chosen center is X_4, then the central quadrilateral has the same area as the reference quadrilateral (Figure 39).*
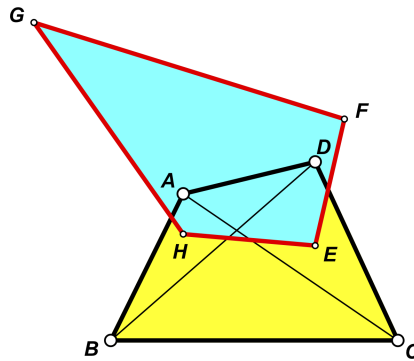

FIGURE 39. orthocenters $\implies [ABCD] = [EFGH]$

Open Question 3. *Is there a purely geometric proof of Theorem 6.4? (Figure 39)*

Conjecture 6.5. *If the reference quadrilateral is a general quadrilateral, $ABCD$, then X_2 is the only center such that the central quadrilateral is similar to the reference quadrilateral for all quadrilaterals $ABCD$.*

Conjecture 6.6. *If the reference quadrilateral is a general quadrilateral, $ABCD$, then X_4 is the only center such that the central quadrilateral has the same area as the reference quadrilateral for all quadrilaterals $ABCD$.*

6.2. **Cyclic Quadrilaterals.**

Theorem 6.7. *If the reference quadrilateral is cyclic, then the central quadrilateral is a rectangle when the chosen center is X_1.*

Proof. This theorem is the same as Result 1 in the Introduction. A proof can be found in [1, p. 133].   □

Theorem 6.8. *If the reference quadrilateral is cyclic, then the central quadrilateral is a rectangle when the center function is of the form*

$$\cos B + \cos C + k \cos A - 1$$

where k is a constant.

The Bevan point of a triangle has center function $\cos B + \cos C - \cos A - 1$. This gives us the following corollary.
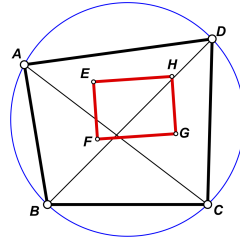
FIGURE 40. Bevan points $\implies$ rectangle

Corollary 6.9. *Let $ABCD$ be a cyclic quadrilateral. Let E, F, G, and H be the Bevan points of triangles $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then $EFGH$ is a rectangle. (Figure 40)*

Open Question 4. *Is there a purely geometric proof of Theorem 6.9? (Figure 40)*

Theorem 6.10. *If the reference quadrilateral is cyclic, then the central quadrilateral degenerates to a line segment when the chosen center is X_{155}. (Figure 41)*
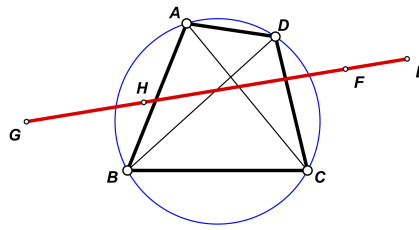

FIGURE 41. X_{155} points $\implies$ line

Theorem 6.11. *If the reference quadrilateral is cyclic, then the central quadrilateral is cyclic when the center function is of the form*

$$\cos B \cos C + k \cos A$$

where k is a constant.

Theorem 6.12. *If the reference quadrilateral is cyclic, then the central quadrilateral is cyclic for the following common centers: the centroid, the orthocenter, the nine-point center, the first and second Fermat points, the first and second isodynamic points, the de Longchamps point, the Far-Out point, the Tarry point, and the Steiner point.*

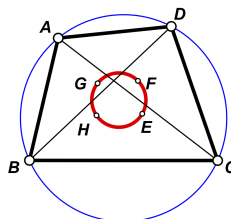One case of Theorem 6.12 is shown in Figure 42.

FIGURE 42. 1st isodynamic points $\implies$ circle

Note. When the center is the orthocenter, the central quadrilateral is congruent to the reference quadrilateral (Figure 43). See [10, p. 209] for a proof.
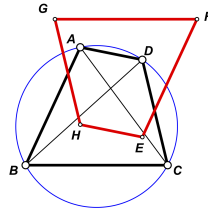

FIGURE 43. orthocenters $\implies$ congruent quadrilaterals

Open Question 5. *Are there purely geometric proofs for the results given in Theorem 6.12?*

The centroid case follows from Theorem 6.3.

Although Theorem 6.12 can be proven using barycentric coordinates, it is also possible to prove the results using complex numbers. We give an example using X_{20}, the de Longchamps point. We start with a Lemma from [10, p. 162].

Lemma 6.13. *The four complex numbers p, q, r, and s are the consecutive vertices of a cyclic quadrilateral (or are collinear) in the complex plane if and only if*

$$\frac{(p-s)(r-q)}{(p-q)(r-s)} \in \mathbb{R}.$$

Theorem 6.14. *If the reference quadrilateral is cyclic, then the central quadrilateral is cyclic if the chosen center is the de Longchamps point.*

Proof. Assume that the circumcircle O of the quadrilateral $ABCD$ is the origin of the complex plane. Let a, b, c, d be the complex coordinates (affixes) of the vertices A, B, C, D. Since $ABCD$ is cyclic we have that the cross ratio

$$(a, b, c, d) = \frac{(a-b)(c-d)}{(a-d)(c-b)}$$

is a real number. The complex coordinates f, g, h, i of the orthocenters F, G, H, I of the half triangles $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, $\triangle ABC$ are ([10, p. 122])

$$f = b + c + d$$
$$g = a + c + d$$
$$h = a + b + d$$
$$i = a + b + c.$$

Since X_{20}, the de Longchamps point, is the reflection of X_4 in $X_3 = 0$, we have

$$f = -(b + c + d)$$
$$g = -(a + c + d)$$
$$h = -(a + b + d)$$
$$i = -(a + b + c).$$

A little calculation shows that

$$(f, g, h, i) = (a, b, c, d) \in \mathbb{R},$$

so f, g, h, i are concyclic. $\qquad\square$

Open Question 6. *Characterize those centers that yield cyclic central quadrilaterals when the reference quadrilateral is cyclic.*

6.3. EqualProdOpp Quadrilaterals.

Theorem 6.15. *If the reference quadrilateral is equalProdOpp, then the central quadrilateral is also equalProdOpp for the following centers: X_3, X_5, X_{15}.*

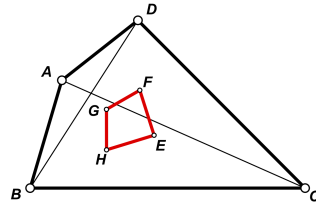This theorem is illustrated in Figure 44 using 1st isodynamic points (X_{15}).

FIGURE 44. X_{15} centers, $AD \cdot BC = AB \cdot CD \implies EF \cdot GH = FG \cdot HE$

Open Question 7. *Are there purely geometric proofs for the results given in Theorem 6.15?*

6.4. Orthodiagonal Quadrilaterals.

Theorem 6.16. *If the reference quadrilateral is orthodiagonal, the central quadrilateral is orthodiagonal whenever the chosen center has center function of the form*

$$\cos B \cos C + k \cos A$$

for some constant k. In this case, the diagonals of $EFGH$ are parallel to the diagonals of $ABCD$.

6.5. Trapezoids.

Theorem 6.17. *If the reference quadrilateral is a trapezoid, the central quadrilateral is a trapezoid whenever the chosen center has center function of the form*

$$\cos B \cos C + k \cos A$$

for some constant k.

This theorem is illustrated in Figure 45 using de Longchamps points (X_{20}), which have center function $\cos B \cos C - \cos A$.
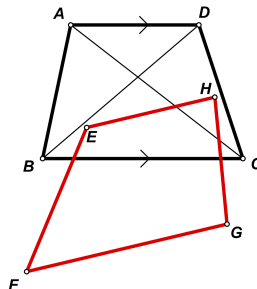

FIGURE 45. de Longchamps points: $AD \parallel BC \implies EH \parallel FG$

6.6. **Tangential Quadrilaterals.**

Theorem 6.18. *If the reference quadrilateral is tangential, then the central quadrilateral is also tangential for the following centers: X_3, X_5.*

This theorem is illustrated in Figure 46 using circumcenters.
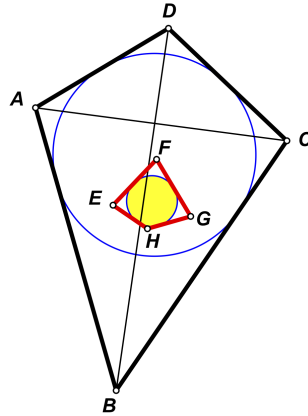

FIGURE 46. Tangential quadrilateral: circumcenters $\implies$ tangential

Open Question 8. *Are there purely geometric proofs for the results given in Theorem 6.18?*

6.7. **Equidiagonal Orthodiagonal Quadrilaterals.**

Theorem 6.19. *If the reference quadrilateral is equidiagonal and orthodiagonal, then the central quadrilateral is also equidiagonal and orthodiagonal for the following centers: X_{51}, X_{373}.*

Theorem 6.20. *If the reference quadrilateral is equidiagonal and orthodiagonal, then the central quadrilateral degenerates to a line segment when the center is X_{642}.*

Theorem 6.21. *If the reference quadrilateral is equidiagonal and orthodiagonal, then the central quadrilateral is orthodiagonal (but not equidiagonal) when the center is the inner Vecten point, X_{486} (Figure 47).*
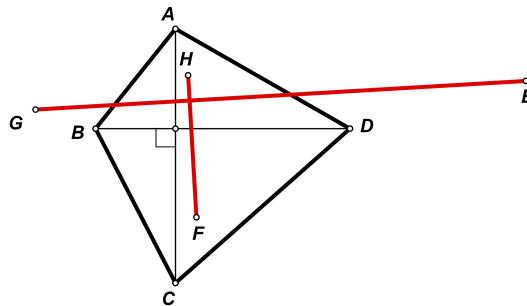

FIGURE 47. inner Vecten pts., $AC = BD, AC \perp BD \implies FH \perp EG$

6.8. **Isosceles Trapezoids.**

Theorem 6.22. *If the reference quadrilateral is an isosceles trapezoid, then the central quadrilateral is also an isosceles trapezoid (Figure 48).*
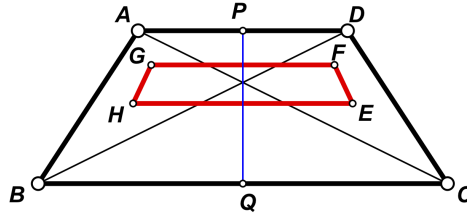

FIGURE 48. isosceles trapezoid $\implies$ isosceles trapezoid

Proof. Let P and Q be the midpoints of AD and BC. Note that triangles $\triangle ABC$ and $\triangle DCB$ are congruent. One is the reflection of the other about PQ. Under this reflection, H maps to E, so $HE \perp PQ$ which means $HE \parallel BC$. Similarly, $FG \parallel BC$. Thus, $EFGH$ is a trapezoid. Segment EF maps to HG under this reflection. Since reflection preserves distances, $EF = HG$. Therefore, $EFGH$ is an isosceles trapezoid. $\qquad\square$

Theorem 6.23. *If the reference quadrilateral is an isosceles trapezoid, then the central quadrilateral is a rectangle for the following centers: X_{40}, X_{165}.*

6.9. **Cyclic Orthodiagonal Quadrilaterals.**

Theorem 6.24. *If the reference quadrilateral is cyclic and orthodiagonal, then the central quadrilateral degenerates to a line segment when the center is X_{63}. (Figure 49)*
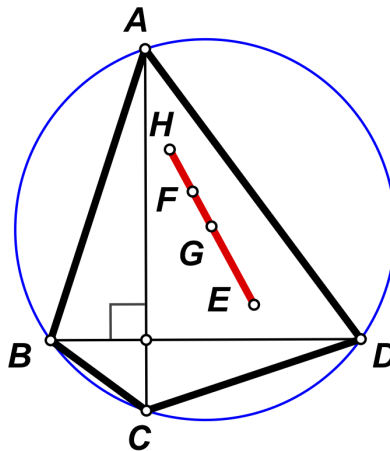

FIGURE 49. X_{63} points $\implies$ line

6.10. Hjelmslev Quadrilaterals.

Theorem 6.25. *If the reference quadrilateral is Hjelmslev, then the central quadrilateral is a parallelogram when the center is X_{53}.*

Theorem 6.26. *If the reference quadrilateral is Hjelmslev, then the central quadrilateral degenerates to a line segment when the center is X_{97}.*

6.11. Kites.

Theorem 6.27. *If the reference quadrilateral is a kite, then the central quadrilateral is a (not necessarily convex) kite (Figure 50).*
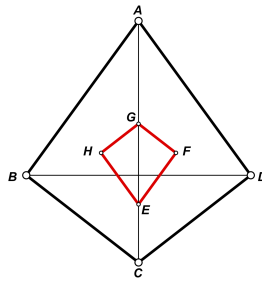
FIGURE 50. kite $\implies$ kite

Proof. Label the kite so that $AB = AD$ and $CB = CD$ (Figure 51).
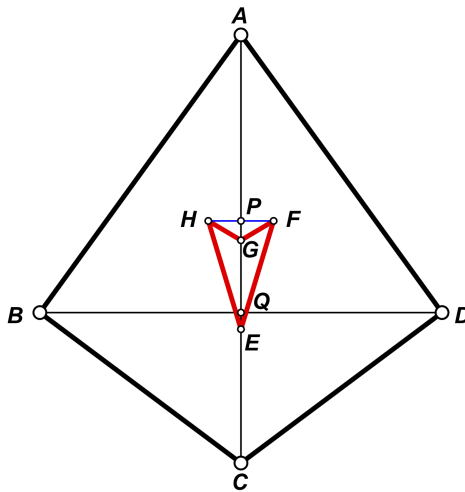
FIGURE 51. central kite need not be convex

Let AC meet BD at Q. The diagonals of a kite are perpendicular, so $AC \perp BD$. A center of an isosceles triangle must lie on the altitude to the base, so centers E and G lie on AC. Let FH meet AC at P. Note that P can lie anywhere on the line through A and C, and is not necessarily located as shown in Figure 51. Note also that quadrilateral $EFGH$ could be convex as shown in Figure 50 or nonconvex as shown in Figure 51.

By the Isosceles Triangle Lemma (Lemma 5.47), $FH \perp AC$ and $PH = PF$. Thus $\triangle PEH \cong \triangle PEF$, so $EH = EF$. Similarly, $\triangle PGH \cong \triangle PGF$, so $GH = GF$. Thus, quadrilateral $EFGH$ is a kite. $\square$

6.12. **Parallelograms.**

Theorem 6.28. *If the reference quadrilateral is a parallelogram, then the central quadrilateral is also a parallelogram (Figure 52).*

FIGURE 52. parallelogram $\implies$ parallelogram

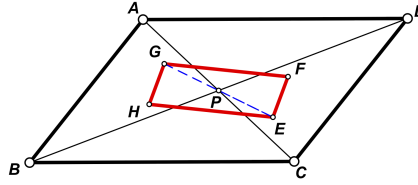*Proof.* Let P be the intersection of the diagonals of the reference quadrilateral $ABCD$. Note that triangles ABD and CDB are congruent. Under the congruence transformation that maps $\triangle ABD$ into $\triangle CDB$, E will get mapped into G. Since P is the center of the congruence transformation, this means EG will pass through P. Congruence preserves lengths, so $EP = PG$ (Figure 52). Similarly, $HP = PF$. Since the diagonals of quadrilateral $EFGH$ bisect each other, this means that quadrilateral $EFGH$ is a parallelogram. $\square$

Theorem 6.29. *If the reference quadrilateral is a parallelogram, then the central quadrilateral is a rhombus when the center is X_{10}, X_{639}, X_{640}, X_{641}, or X_{642}. (Figure 53 shows the case for X_{10})*

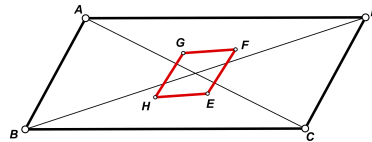FIGURE 53. parallelogram and X_{10} points $\implies$ rhombus

Open Question 9. *Is there a purely geometrical proof to the special case of Theorem 6.29 shown in Figure 53 where the chosen center is the Spieker center?*

6.13. **Rhombi.**

Theorem 6.30. *If the reference quadrilateral is a rhombus, then the central quadrilateral is also a rhombus (Figure 54).*
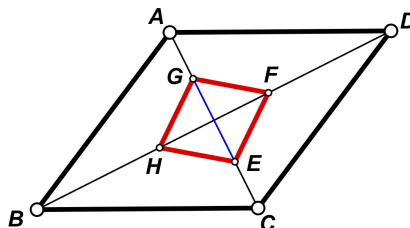
FIGURE 54. rhombus $\implies$ rhombus

Proof. A center of an isosceles triangle must lie on the altitude to the base. Thus, E and G lie on AC and F and H lie on BD. Thus, the diagonals of $EFGH$ lie along the diagonals of $ABCD$. The diagonals of $ABCD$ are perpendicular, so the diagonals of $EFGH$ are also perpendicular. This, combined with Theorem 6.28, implies that $EFGH$ is a rhombus. □

Theorem 6.31. *If the reference quadrilateral is a rhombus, then the central quadrilateral is a square when the center is the Weill point (X_{354}).*

6.14. **Rectangles.**

Theorem 6.32. *If the reference quadrilateral is a rectangle, then the central quadrilateral is a rectangle (Figure 55).*

FIGURE 55. rectangle $\implies$ rectangle

Proof. Note that triangles $\triangle ABC$ and $\triangle DCB$ are congruent. Under the reflection that maps $\triangle ABC$ into $\triangle DCB$, H will get mapped into E and HE will be parallel to BC. Similarly, $GF \parallel AD$, $GH \parallel AB$, and $FE \parallel DC$. Thus, $EFGH$ is a rectangle. □

6.15. **Squares.**

Theorem 6.33. *If the reference quadrilateral is a square, then the central quadrilateral is a square (Figure 56).*

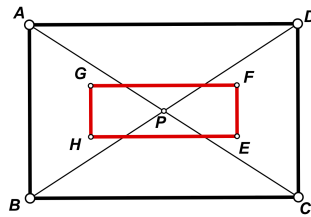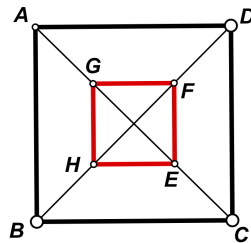FIGURE 56. square $\implies$ square

Proof. By Theorem 6.28, $EFGH$ must be a parallelogram. By Theorem 6.27, $EFGH$ must be a kite. But any parallelogram that is also a kite must be a square. □

7. Areas for future research

There are many avenues for future investigation.

Generalizing the center.

Instead of placing a triangle center in each component triangle, there may be other types of points that can be used. For example, consider the following result which comes from [12]. See also [6] for generalizations.

Theorem 7.1. *Let $ABCD$ be an equidiagonal quadrilateral with diagonal point E. Locate points F, G, H, and I inside triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, so that triangles $\triangle ABF$, $\triangle BCG$, $\triangle CDH$, and $\triangle DAI$ are similar isosceles triangles. Then $FGHI$ is an orthodiagonal quadrilateral. (Figure 57)*
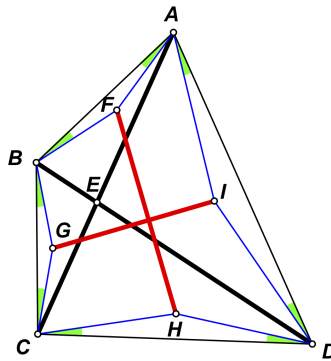
Figure 57. $AC = BD \implies FH \perp GI$

Using different component triangles.

There are many ways of forming four component triangles from a given quadrilateral besides using quarter triangles or half triangles. For example, if the quadrilateral is $ABCD$, we could pick a point P inside the quadrilateral and then consider the four component triangles $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$.

For example consider the following result due to Dao [2].

Theorem 7.2. *Let $ABCD$ be a tangential quadrilateral. Let E be a point inside the quadrilateral such that $AE = CE$ and $BE = DE$. Let F, G, H, and I be the incenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a cyclic quadrilateral. (Figure 58)*
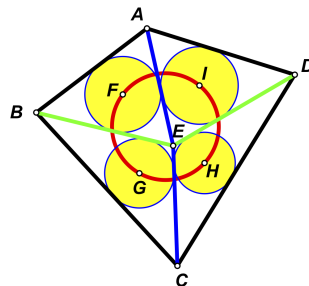
Figure 58. $ABCD$ tangential, $AE{=}CE$, $BE{=}DE \implies FGHI$ concyclic

There are many other interesting points associated with a quadrilateral. For example, we could specify that E is the centroid, Poncelet point, or Steiner point of the quadrilateral. If the quadrilateral is cyclic, we could specify that E is the anticenter.

We could even ask if there are any properties when E is an arbitrary point inside the quadrilateral. For example, we have the following result found by computer.

Theorem 7.3. *Let $ABCD$ be an equidiagonal quadrilateral. Let E be an arbitrary point inside the quadrilateral. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a rhombus. (Figure 59)*
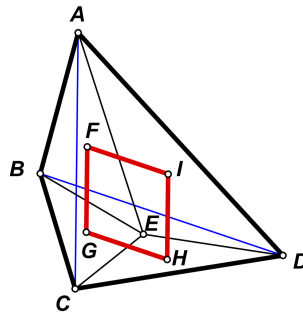
FIGURE 59. Equidiagonal quadrilateral, centroids $\implies$ rhombus

Another possibility is to use the four triangles determined by the sides of the reference quadrilateral taken three at a time.

Using different centers.

Instead of placing the same center in each of the four component triangles, we might try placing different centers in each triangle, or different centers in each pair of triangles. For example, the following result was found by computer.

Theorem 7.4. *Let $ABCD$ be a tangential quadrilateral. Let E be the incenter of $\triangle BCD$. Let F be the centroid of $\triangle ACD$. Let G be the incenter of $\triangle ABD$. Let H be the centroid of $\triangle ABC$. Then $EFGH$ is an orthodiagonal quadrilateral (Figure 60).*
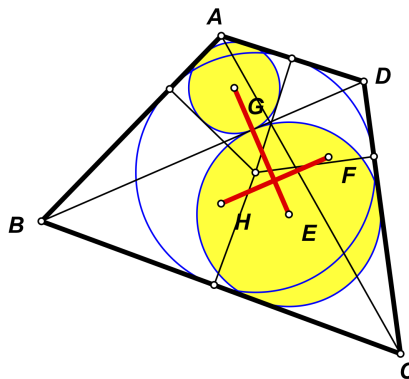
FIGURE 60. Tangential quad., incenters/centroids $\implies$ $EG \perp FH$

Checking for other properties.

After we have placed points inside four component triangles, we can ask other questions about the central quadrilateral besides asking about its shape. We can compare the central quadrilateral with the reference quadrilateral and ask questions like "are they similar?", "are they congruent?", "do they have the same centroid?", "are they homothetic?", "are they in perspective?", "do they have the same area?", etc.

Here is an example found by computer.

Theorem 7.5. *Let ABCD be a rhombus with diagonal point E. Let F, G, H and I be the Bevan point (X_{40}) of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then quadrilaterals ABCD and FGHI have the same area and perimeter (Figure 61).*

FIGURE 61. Rhombus: Bevan points $\implies$ same area and perimeter

We could also ask if a certain center of the reference quadrilateral coincides with a center of the central quadrilateral. Here, "center" refers to some well-known point associated with a quadrilateral, such as the centroid, Poncelet point, Steiner point, anticenter, etc.

There are lots of interesting results still waiting to be discovered!

Appendix A. Raw Data for Quarter Triangles

Theorem A.1. *For a general quadrilateral,*
(a) The central quadrilateral is orthodiagonal if $n = 1$.
(b) The central quadrilateral is a parallelogram if $n =$2-5, 20, 140, 376, 381-382, 546-550, 631-632.

No additional results were obtained for the following types of quadrilaterals:

- *APquad*
- *bicentric*
- *equalProdAdj*
- *equalProdOpp*
- *exbicentric*
- *extangential*
- *harmonic*
- *Pythagorean*
- *tangential trapezoid*
- *trapezoid*

Theorem A.2. *For a cyclic quadrilateral,*
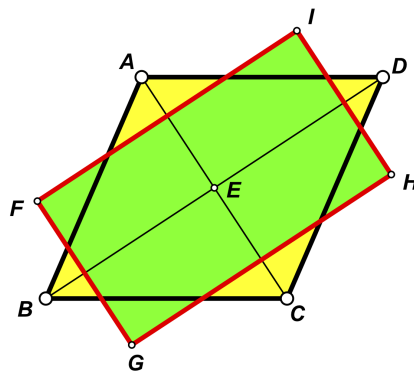*(excluding results for a general quadrilateral)*
(a) The central quadrilateral is an isosceles trapezoid if $n = 110$.
(b) The central quadrilateral is orthodiagonal if $n =$485-486.

Theorem A.3. *For an equidiagonal quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is orthodiagonal if $n =$8, 10, 40, 165, 355, 551, 946, 962.
(b) The central quadrilateral is a rhombus if $n =$2-5, 20, 140, 376, 381-382, 546-550, 631-632.

Theorem A.4. *For an orthodiagonal quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is orthodiagonal if $n =$46-48, 73, 90-91, 163, 223, 226, 282, 284, 336, 380, 388, 485-486, 493-494, 497, 563, 579-581, 610, 652, 656, 820, 822, 836, 920-921, 944-951.
(b) The central quadrilateral is equidiagonal if $n =$151.
(c) The central quadrilateral is cyclic if $n =$6, 51, 54, 67, 70, 74, 125, 130, 184-185, 217, 287-288, 296, 389, 578, 686, 973-974.
(d) The central quadrilateral is equidiagonal, orthodiagonal if $n =$102, 109, 117, 124.
(e) The central quadrilateral is a rectangle if $n =$2, 3, 5, 20 , 22-23, 95, 97, 122-123, 127, 131, 140, 175, 216, 233, 253, 268, 280, 339, 347, 376, 381-382, 401-402, 408, 417-418, 426, 440, 441, 454, 464-466, 546-550, 577, 631-632, 828, 852, 856, 858, 925.
(f) The central quadrilateral degenerates to a point for all centers that occur at the vertex of the right angle in a right triangle.

Theorem A.5. *For an isosceles trapezoid,*
(excluding results for an equidiagonal quadrilateral)
(excluding results for a cyclic quadrilateral)
(excluding results for trapezoid)

(a) The central quadrilateral is a kite if n =1, 6, 8-12, 14,15, 17-19, 21-22, 29, 33, 35, 37-42,45, 47-48 and many more.

Theorem A.6. *For an orthodiagonal trapezoid,*
(excluding results for a trapezoid)
(excluding results for an orthodiagonal quadrilateral)
(a) The central quadrilateral is orthodiagonal if n = 1, 48, 73, 223, 226, 282, 284, 336, 371, 380, 388, 485, 493, 497, 581, 610, 820, 836, 944-951.
(b) The central quadrilateral is an equidiagonal quadrilateral if n = 151.
(c) The central quadrilateral is Hjelmslev if n =6, 51, 54, 67, 70, 74, 125, 130, 184-185, 217, 287-288, 296, 389, 578, 686, 973-974.
(d) The central quadrilateral is an equidiagonal, orthodiagonal quadrilateral if n =102, 124.

Theorem A.7. *For a parallelogram,*
(excluding results for a trapezoid)
(excluding results for Pythagorean)
(excluding results for extangential)
(a) The central quadrilateral is a parallelogram for all n.

Theorem A.8. *For a tangential quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is cyclic orthodiagonal if n = 1.

Theorem A.9. *For a Hjelmslev quadrilateral,*
(excluding results for a cyclic quadrilateral)
(excluding results for a Pythagorean quadrilateral)
(a) The central quadrilateral is orthodiagonal if n =48, 91, 563.
(b) The central quadrilateral is a trapezoid if n =68, 155, 317, 577.

Theorem A.10. *For a kite,*
(excluding results for an orthodiagonal quadrilateral)
(excluding results for a tangential quadrilateral)
(excluding results for an extangential quadrilateral)
(excluding results for a Pythagorean quadrilateral)
(excluding results for an equalProdOpp quadrilateral)
(excluding results for an equalProdAdj quadrilateral)
(a) The central quadrilateral is an equidiagonal orthodiagonal trapezoid if n =1, 46-48, 73, 90-91, 102, 109, 117, 124, 163, 223, 226, 282, 284, 336, 380, 388, 485-486, 493-494, 497, 563-564, 579-581, 610, 652, 656, 820, 822, 836, 920-921, 944-951.
(b) The central quadrilateral is an isosceles trapezoid if n=6-18, and many more.
(c) The central quadrilateral is a rectangle if n =2, 3, 5, 20, 22-23, 95, 97, 122-123, 127, 131, 140, 175, 216, 233, 253, 268, 280, 339, 347, 376, 381-382, 401-402, 408, 417-418, 426, 440-441, 454, 464-466, 546-550, 577, 631-632, 828, 852, 856, 858, 925.

Theorem A.11. *For a rhombus,*
(excluding results for a parallelogram)
(excluding results for a tangential trapezoid)
(excluding results for a kite)
(excluding results for an orthogonal trapezoid)
(a) The central quadrilateral is a square if n =1, 46-48, 73, 90-91, 102, 109, 117,

124, 163, 223, 226, 282, 284, 336, 380, 388, 485-486, 493-494, 497, 563, 564, 579-581, 610, 652, 656, 820, 822, 836, 920-921, 944-951.
(b) The central quadrilateral is a rectangle if n=2-3, 5-18, 20-23, 31-32, 35-52, 54-63, 65, 67, 69-89, 94-101, and many more.

Theorem A.12. *For an equidiagonal orthodiagonal quadrilateral,*
(excluding results for an equidiagonal quadrilateral)
(excluding results for an orthodiagonal quadrilateral)
(a) The central quadrilateral is orthodiagonal if n =1, 8, 10, 40, 46-48, 63, 73, 84, 90-91, 145, 163, 165, 189, 197, 223, 226-227, 255, 271, 282-284, 307, 336, 355, 380, 388, 485, 488, 493, 497, 551, 581, 610, 637, 639, 641, 820, 836, 944-951, 958, 962, 993.
(b) The central quadrilateral is a square if n =2, 3, 5, 20,22-23, 95, 97, 122-123, 127, 131, 140, 175, 216, 233, 253, 268, 280, 339, 347, 376, 381-382, 401-402, 408, 417-418, 426, 440-441, 454, 464-466, 546-550, 577, 631-632, 828, 852, 856, 858, 925.
(c) The central quadrilateral is cyclic if n =6, 51, 54, 67, 70, 74, 125, 130, 184-185, 217, 287-288, 296, 389, 578, 686, 973-974.
(d) The central quadrilateral is equidiagonal, orthodiagonal if n =102, 124, 151.

Theorem A.13. *For an equidiagonal orthodiagonal kite,*
(excluding results for an equidiagonal orthodiagonal quadrilateral)
(excluding results for a kite)
(a) The central quadrilateral is an equidiagonal, orthodiagonal trapezoid if n =1, 8, 10, 40, 46-48, 63, 73, 84, 90-91, 102, 124, 145, 151, 165, 189, 197, 223, 226-227, 255, 271, 282-284, 307, 336, 355, 380, 388, 485, 488, 493, 497, 551, 581, 610, 637, 639, 641, 820, 836, 944-951, 958, 962, 993.
(b) The central quadrilateral is a square if n =2, 3, 5, 20, 22-23, 95, 97, 122-123, 127, 131, 140, 175, 216, 233, 253, 268, 280, 339, 347, 376, 381-382, 401-402, 408, 417-418, 426, 440-441, 454, 464-466, 546-550, 577, 631-632, 828, 852, 856, 858, 925.
(c) The central quadrilateral is an isosceles trapezoid if n =6-7, 9, 11-18, 21, 31-32, and many more values (almost all n).
(d) The central quadrilateral is a rectangle if n =224, 487, 914.
(e) The central quadrilateral is a line segment if n =109, 117, 663.

Theorem A.14. *For an equidiagonal orthodiagonal trapezoid,*
(excluding results for an equidiagonal orthodiagonal quadrilateral)
(excluding results for an orthogonal trapezoid)
(excluding results for an isosceles trapezoid)
(a) The central quadrilateral is an equiOrthoKite if n =102, 124, 151, 638, 640.
(b) The central quadrilateral is a kite if n =1, 7-17, 21, 31, 35-43, 44-45, 48, 55, 57-58, 60-63, 65, 69, 71-73, 75-83, 85, 86, 88-89, 94, 98, 103-106, and many more.
(c) The central quadrilateral is bicentric, exbicentric, harmonic, and Hjelmslev if n =6, 51, 54, 67, 70, 74, 125, 130, 184-185, 217, 287-288, 296, 389, 578, 973-974.
(d) The central quadrilateral is a square if n =2, 3, 5, 20, 22-23, 95, 97, 122-123, 127, 140, 175, 216, 233, 253, 268, 280, 339, 347, 376, 381-382, 401-402, 408, 417-418, 426, 440-441, 464-466, 546-550, 577, 631-632, 828, 852, 856, 858.

Theorem A.15. *For a rectangle,*
(excluding results for a harmonic quadrilateral)
(excluding results for a Hjelmslev quadrilateral
(excluding results for an isosceles trapezoid)
(excluding results for a parallelogram)
(a) The central quadrilateral is a rhombus for all triangle centers.

Theorem A.16. *For a square,*
(excluding results for an equidiagonal orthogonal trapezoid)
(excluding results for an equidiagonal orthogonal kite)
(excluding results for a cyclic orthodiagonal quadrilateral)
(excluding results for a bicentric trapezoid)
(excluding results for a rhombus)
(excluding results for a rectangle)
(excluding results for an exbicentric quadrilateral)
(a) The central quadrilateral is a square for all triangle centers.

Appendix B. Raw Data for Half Triangles

Theorem B.1. *For a general quadrilateral,*
no results were found.

Theorem B.2. *For a cyclic quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is a rectangle for n =1, 40, 165.
(b) The central quadrilateral is a line segment for n =155.
(c) EFGH is cyclic for n =2, 4, 5, 13-16, 20, 23, 26, 36, 74, 80, 98-100
and a large number of other n.
Missing: 6-12, 17-19, 21, 22, 24, 25, 27-29...

Theorem B.3. *For an equalProdOpp quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is equalProd for n =2, 3, 5, 15.

Theorem B.4. *For an equalProdAdj quadrilateral,*
(excluding results for a general quadrilateral)
no new results were found.

Theorem B.5. *For an equidiagonal quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is equidiagonal for n =2.

Theorem B.6. *For an extangential quadrilateral,*
(excluding results for general quadrilateral)
(a) The central quadrilateral is extangential for n =2, 3, 5.

Theorem B.7. *For an orthodiagonal quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is orthodiagonal for n =2-5, 20, 51, 140, 376, 381,
382, 546-550, 631-632.

Theorem B.8. *For a Pythagorean quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is Pythagorean for n =2.

Theorem B.9. *For a trapezoid,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is a trapezoid for n =2-5, 20, 140, 376, 381, 382, 546-550, 631-632.

Theorem B.10. *For a tangential quadrilateral,*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is tangential for n =2, 3, 5.

Theorem B.11. *For an APquad,*
(excluding results for an extangential quadrilateral)
(a) The central quadrilateral is an APquad for n =2.

Theorem B.12. *For an equidiagonal orthodiagonal quadrilateral,*
(excluding results for an equidiagonal quadrilateral)
(excluding results for an orthodiagonal quadrilateral)
(a) The central quadrilateral is an equidiagonal orthodiagonal quadrilateral for n =2, 51, 373.
(b) The central quadrilateral is a line segment for n =642.
(c) The central quadrilateral is orthodiagonal for n =486.

Theorem B.13. *For an orthodiagonal trapezoid,*
(excluding results for a trapezoid)
(excluding results for an orthodiagonal quadrilateral)
(a) The central quadrilateral is orthodiagonal for n =2-5, 20, 140, 376, 381, 382, 546-550, 631-632.
(b) The central quadrilateral is orthodiagonal for n =51-53, 143.

Theorem B.14. *For an isosceles trapezoid,*
(excluding results for an equidiagonal quadrilateral)
(excluding results for a cyclic quadrilateral)
(excluding results for trapezoid)
(a) The central quadrilateral is a rectangle for n =1, 40, 165.
(b) The central quadrilateral appears to be an isosceles trapezoid for just about all other n.

Theorem B.15. *For a cyclic orthodiagonal quadrilateral,*
(excluding results for a cyclic quadrilateral)
(excluding results for an orthodiagonal quadrilateral)
(a) The central quadrilateral is an isosceles trapezoid for n =26.
(b) The central quadrilateral is an isosceles trapezoid for n =53, 143, 373.
(c) The central quadrilateral is a line segment for n =63.
(d) The central quadrilateral is a trapezoid for n =577.

Theorem B.16. *For a tangential trapezoid,*
(excluding results for a tangential quadrilateral)
(excluding results for a trapezoid)
(a) The central quadrilateral is a tangential trapezoid for n =2, 3, 5.

Theorem B.17. *For a tangential trapezoid, ?????*
(excluding results for a general quadrilateral)
(a) The central quadrilateral is orthodiagonal for n =2-5, 20, 140, 376, 381, 382, 546-550, 631-632.
(b) The central quadrilateral is cyclic for n =2-5, 20, 140, 376, 381, 382,

Theorem B.18. *For a kite,*
(excluding results for an orthodiagonal quadrilateral)
(excluding results for a tangential quadrilateral)
(excluding results for an extangential quadrilateral)
(excluding results for a Pythagorean quadrilateral)
(excluding results for an equalProdOpp quadrilateral)
(excluding results for an equalProdAdj quadrilateral)
(a) The central quadrilateral is a kite for all n.

Theorem B.19. *For a bicentric quadrilateral,*
(excluding results for a tangential quadrilateral)
(excluding results for cyclic quadrilateral)
(a) The central quadrilateral is bicentric for n =2, 4, 5, 20, 140, 376, 381, 382, 546-550, 631, 632.

Theorem B.20. *For a harmonic quadrilateral,*
(excluding results for a cyclic quadrilateral)
(excluding results for an equalProdOpp quadrilateral)
(a) The central quadrilateral is harmonic for n =2, 4, 5, 15, 16, 20, 23, 125, 140, 186, 265, 376, 381, 382, 546-550, 631, 632.

Theorem B.21. *For a parallelogram,*
(excluding results for Pythagorean)
(excluding results for extangential)
(a) The central quadrilateral is a rhombus for n =10, 639-642.
(b) The central quadrilateral appears to be a parallelogram for all other n.

Theorem B.22. *For a Hjelmslev quadrilateral,*
(excluding results for a cyclic quadrilateral)
(excluding results for a Pythagorean quadrilateral)
(a) The central quadrilateral is Hjelmslev for n =2, 4, 5, 20, 102, 109, 140, 376, 381, 382, 546-550, 631, 632, 930.
(b) The central quadrilateral is a parallelogram for n =53.
(c) The central quadrilateral is a line segment for n =97.

Theorem B.23. *For an exbicentric quadrilateral,*
(excluding results for an extangential quadrilateral)
(excluding results for a cyclic quadrilateral)
(a) The central quadrilateral is exBicentric for n =2, 4, 5, 20, 140, 376, 381, 382, 546-550, 631, 632.

Theorem B.24. *For a bicentric trapezoid,*
(excluding results for an isosceles quadrilateral)
(excluding results for an tangential quadrilateral)
(excluding results for a bicentric quadrilateral)
(a) The central quadrilateral is a bicentric trapezoid for n =2, 4, 5, 20, 140, 376, 381, 382, 546-550, 631, 632.
(b) The central quadrilateral appears to be an isosceles trapezoid for just about all other n.

Theorem B.25. *For a rhombus,*
(excluding results for a parallelogram)
(excluding results for a tangential trapezoid)

(excluding results for a kite)
(excluding results for an orthogonal trapezoid)
(a) The central quadrilateral is a rhombus for all n.
(b) The central quadrilateral is a square for n =354.

Theorem B.26. *For an equidiagonal orthodiagonal kite,*
(excluding results for an equidiagonal orthodiagonal quadrilateral)
(excluding results for a kite)
(a) The central quadrilateral is a bicentric exbicentric harmonic Hjelmslev quadrilateral for n =620.
(b) The central quadrilateral is an equidiagonal orthodiagonal kite for n =2, 51, 373, 638.
(c) The central quadrilateral is a line segment for n =615.

Theorem B.27. *For an equidiagonal orthodiagonal trapezoid,*
(excluding results for an equidiagonal orthodiagonal quadrilateral)
(excluding results for an orthogonal trapezoid)
(excluding results for an isosceles trapezoid)
(a) The central quadrilateral is a bicentric trapezoid for n =157.
(b) The central quadrilateral is cyclic for n =70, 565, 657, 770.
(c) The central quadrilateral is an equidiagonal orthodiagonal trapezoid for n =2, 4, 5, 20, 51-53, 68, 91, 96, 135, 137, 140, 143, 373, 376, 381, 382, 389, 486, 546-550, 568, 571, 631, 632, 847, 925.
(d) The central quadrilateral is a rectangle for n =1, 26, 40, 131, 165, 577, 578.
(e) The central quadrilateral is a trapezoid for n =70, 565, 657, 770.
(f) The central quadrilateral is a line segment for n =63, 136, 494, 642.
(g) The central quadrilateral appears to be an isosceles trapezoid for most other n.

Theorem B.28. *For a rectangle,*
(excluding results for a harmonic quadrilateral)
(excluding results for a Hjelmslev quadrilateral
(excluding results for an isosceles trapezoid)
(excluding results for a parallelogram)
(a) The central quadrilateral is a point for n =3, 97, 122, 123, 127, 131, 216, 268, 339, 408, 417, 418, 426, 440, 441, 454, 464-466, 577, 828, 852, 856.
(b) The central quadrilateral is a square for n =10, 117, 124, 197, 227, 355, 639-642, 958, 993.
(c) The central quadrilateral appears to be rectangle for most other n.

Theorem B.29. *For a square,*
(excluding results for an equidiagonal orthogonal trapezoid, an equidiagonal orthogonal kite, a cyclic orthodiagonal quadrilateral, a bicentric trapezoid, an exbicentric quadrilateral, a rhombus, and a rectangle)
(a) The central quadrilateral is a point for n =3, 11, 48, 49, 63, 69, 71-73, 77, 78, 97, 115, 116, 122-125, 127, 130, 137, 184, 185, 201, 212, 216, 217, 219, 222, 228, 244-246, 248, 255, 265, 268, 271, 283, 287, 293, 295, 296, 304-307, 326, 328, 332, 336-339, 343, 345, 348, 394, 408, 417, 418, 426, 440, 441, 464-466, 488, 499, 577, 591, 603, 606, 615, 640, 682, 748, 820, 828, 836, 852, 856, 865-868, 895, 974.
(b) The central quadrilateral appears to be a square all other n.

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